The non-isothermal spreading of a thin drop on a heated or cooled horizontal substrate

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Abstract

We re-visit the problem of the spreading of a thin two-dimensional drop of incompressible Newtonian fluid on a uniformly heated or cooled smooth horizontal substrate when thermocapillary effects are significant. The dynamics of the moving contact line are modelled by a "Tanner Law" relating the contact angle to the speed of the contact line. The present work builds on earlier theoretical investigations by Burelbach, Bankoff and Davis [1] and Ehrhard and Davis [2], the latter of whom derived the non-linear partial differential equation governing the free-surface profile of the drop. By adapting the approach used by Holland, Duffy and Wilson [3] we obtain the (implicit) exact solution of the two-dimensional equation in the case of quasi-steady motion. We consider the behaviour of the solution in various asymptotic limits which confirm and extend some results of Ehrhard and Davis [2]. We show that multiple solutions are possible for the case of a pendent drop on an appropriately cooled substrate; the three solutions are shown to be qualitatively different in both shape and flow pattern.

1 Introduction

The spreading of a thin drop is a fundamental problem in fluid mechanics, with a vast number of industrial applications. The review article by Oron, Davis and Bankoff [4] gives an excellent overview of the recent theoretical work done on this and many other thin-film flows. The pioneering work on non-isothermal thin-film flow was by Burelbach, Bankoff and Davis [1] who formulated and analysed the general evolution equation for a two-dimensional thin film of fluid on a uniformly heated or cooled substrate, including the effects of mass loss, vapour recoil, thermocapillarity, surface tension, gravity and long-range inter-molecular attraction. Ehrhard and Davis [2] used a special case of this equation (and its axisymmetric analogue) to study the quasi-steady spreading of both two-dimensional and axisymmetric drops on a uniformly heated or cooled horizontal substrate subject to thermocapillary effects. More recent work by Anderson and Davis [5], Ajaev [6] and Hu and Larson [7] have used a similar approach to study evaporating drops.

In this paper we shall re-visit the two-dimensional problem studied by Ehrhard and Davis [2] and, by adopting the approach used by Holland, Duffy and Wilson [3], obtain the (implicit) exact solution of the ordinary differential equation for the free-surface profile of the drop.

2 Formulation

Consider the quasi-steady spreading of a two-dimensional drop of an incompressible Newtonian fluid with uniform density ρ, viscosity μ, specific heat c and thermal conductivity k_{th} on a smooth horizontal substrate. The velocity $\mathbf{u} = (u, v, w)$, pressure p and temperature T of the fluid are governed by the familiar mass-conservation, Navier-Stokes and energy equations referred to the Cartesian coordinates $Oxyz$ indicated in Figure 1. At the solid substrate $z = 0$ the fluid velocity is zero and the temperature is equal to the prescribed uniform substrate temperature T_0 (different from the prescribed uniform temperature T_∞ of the surrounding vapour). On the free surface $z = h(x, t)$ the appropriate boundary conditions are normal and tangential stress balances, an energy balance and the kinematic condition. We take ρ, μ, c, k_{th} and the unit surface thermal conductance α_{th} to be constants, but we assume that

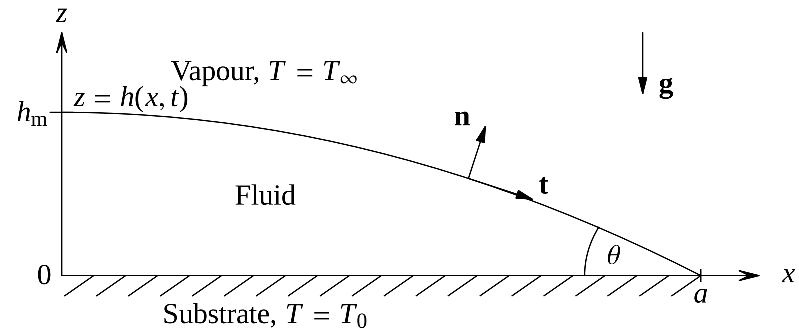

Figure 1: Geometry of the problem.

the surface tension γ depends linearly on temperature according to

$$\gamma(T) = \gamma_0 - \lambda(T - T_0), \tag{1}$$

where $\lambda = -\mathrm{d}\gamma/\mathrm{d}T$ is a positive constant and γ_0 is the constant surface tension at $T = T_0$. We shall consider only solutions that are symmetric about $x = 0$ and smooth at $x = 0$, so that they satisfy

$$h_x = 0, \quad h_{xxx} = 0 \tag{2}$$

at $x = 0$; therefore hereafter we need consider the solution in $0 \le x \le a$ only (with the behaviour in $-a \le x \le 0$ given by symmetry). At the position of the contact line $x = a(t)$ at which $h = 0$ the contact angle takes the value $\theta = \theta(t)$. We shall follow Ehrhard and Davis [2] and assume that the velocity of the contact line depends on the contact angle according to an empirically determined "Tanner Law" in the form

$$a_t = \kappa(\theta - \theta_\infty)^m, \tag{3}$$

where κ is an empirically determined coefficient with dimensions of velocity, θ_∞ is the equilibrium value of θ (which may be zero or non-zero), and m is an odd number, usually 1 or 3. The constant volume of the drop (per unit width in the transverse direction), V, is given by

$$V = 2\int_0^a h \,\mathrm{d}x. \tag{4}$$

In order to make analytical progress we follow many previous authors (notably, Ehrhard and Davis [2]) and consider the case of a thin drop whose cross section is slender (with, in particular, $\theta \ll 1$) and non-dimensionalise the dependent and independent variables as follows:

$$\begin{aligned} &x^* = \sqrt{\frac{\theta_0}{V}}\,x, \quad z^* = \frac{z}{\sqrt{\theta_0 V}}, \quad h^* = \frac{h}{\sqrt{\theta_0 V}}, \quad t^* = \frac{\kappa\theta_0^{m+\frac{1}{2}}t}{\sqrt{V}}, \quad \theta^* = \frac{\theta}{\theta_0}, \\ &u^* = \frac{u}{\kappa\theta_0^m}, \quad w^* = \frac{w}{\kappa\theta_0^{m+1}}, \quad p^* = \frac{\sqrt{V}(p - p_\infty)}{\mu\kappa\theta_0^{m-\frac{3}{2}}}, \quad T^* = \frac{T - T_\infty}{T_0 - T_\infty}, \end{aligned} \tag{5}$$

where $\theta_0 = \theta(0)$; in this scaling the volume of the drop is $V^* = 1$. Note that Ehrhard and Davis [2] used a different non-dimensionalisation involving $a_0 = a(0)$. Moreover, for quasi-steady motion the volume V *cannot* be prescribed independently of $a(0)$ and $\theta(0)$ as was done by Ehrhard and Davis [2]; this oversight in Ehrhard and Davis's [2] analysis was subsequently pointed out and corrected by Ehrhard [8, Appendix]. For simplicity we immediately drop the superscript stars. The leading order equations and boundary conditions can be readily solved, and the pressure and temperature are found to be

$$Cp = G(h - z) - h_{xx}, \quad T = \frac{1 + B(h - z)}{1 + Bh}. \tag{6}$$

The velocity and streamfunction are given by

$$Cu = \frac{Mh_x z(3z - 2h)}{4h(1 + Bh)^2}, \tag{7}$$

$$Cw = \frac{Mz^2}{4h^2(1 + Bh)^3}\left[h(1 + Bh)(h - z)h_{xx} + h_x^2\,(z + Bh(3z - 2h))\right], \tag{8}$$

$$C\psi = \frac{Mh_x z^2(h - z)}{4h(1 + Bh)^2}, \tag{9}$$

where h satisfies the third-order ordinary differential equation

$$(h_{xx} - Gh)_x + \frac{3Mh_x}{2h(1 + Bh)^2} = 0, \tag{10}$$

to be integrated subject to (2), (4) and

$$h = 0, \quad h_x = -\theta \quad \text{at} \quad x = a, \tag{11}$$

and where the non-dimensional capillary, Biot, Bond and Marangoni numbers are given by

$$C = \frac{\mu\kappa}{\gamma_0\theta_0^{3-m}}, \quad B = \frac{\alpha_{\mathrm{th}}\sqrt{\theta_0 V}}{k_{\mathrm{th}}}, \quad G = \frac{\rho g V}{\gamma_0\theta_0}, \quad M = \frac{\alpha_{\mathrm{th}}\sqrt{V}\gamma_0(T_0 - T_\infty)}{k_{\mathrm{th}}\gamma_0\theta_0^{\frac{3}{2}}}, \tag{12}$$

respectively.

3 Implicit Solution of Equation (10)

Holland *et al.* [3, Eq. 36] obtained an equation equivalent to (10) for the transverse profile of a thin rivulet draining steadily down a uniformly heated or cooled substrate in the presence of thermocapillary effects. In their work Holland *et al.* [3] obtained an implicit solution to the equation, and we can readily adapt their solution to the present quasi-steady problem. Specifically, it is found that the solution of (10) may be written in the implicit form

$$x = h_{\mathrm{m}} \int_{h/h_{\mathrm{m}}}^{1} \frac{1}{[F(s)]^{\frac{1}{2}}}\,\mathrm{d}s \tag{13}$$

for $0 \le x \le a$; the constant-volume condition (4) and the condition $h = 0$ when $x = a$ lead to

$$1 = 2h_{\mathrm{m}}^2 \int_0^1 \frac{s}{[F(s)]^{\frac{1}{2}}}\,\mathrm{d}s, \tag{14}$$

$$a = h_{\mathrm{m}} \int_0^1 \frac{1}{[F(s)]^{\frac{1}{2}}}\,\mathrm{d}s, \tag{15}$$

respectively, where h_{m} denotes the maximum height of the drop at $x = 0$, and we have defined

$$F(s) = (1 - s)(\theta^2 - Gh_{\mathrm{m}}^2 s) - 3Mh_{\mathrm{m}} s \log\left(\frac{(1 + Bh_{\mathrm{m}})s}{1 + Bh_{\mathrm{m}} s}\right). \tag{16}$$

Moreover, with (14) equation (15) may be written

$$a = \frac{1}{2h_{\mathrm{m}}} + h_{\mathrm{m}} \int_0^1 \frac{1 - s}{[F(s)]^{\frac{1}{2}}}\,\mathrm{d}s. \tag{17}$$

Equation (14) is an algebraic equation determining h_m, and then (15) determines a explicitly, and (13) determines h implicitly. The evolution of the drop is determined by the Tanner Law

$$a_t = (\theta - \theta_\infty)^m \tag{18}$$

subject to the the initial condition $\theta(0) = 1$. For brevity we henceforth restrict our attention to the limit $B \to 0$.

For later use we note here that, in general, the integrands in (13)–(15) are finite except when $s \to 1$. Expanding F near $s = 1$ yields

$$F(s) = C_1(1 - s) + C_2(1 - s)^2 + O(1 - s)^3 \tag{19}$$

as $s \to 1$, where

$$C_1 = \theta^2 - Gh_\mathrm{m}^2 + 3Mh_\mathrm{m}, \quad C_2 = Gh_\mathrm{m}^2 - \frac{3Mh_\mathrm{m}}{2}. \tag{20}$$

Since F must be positive as $s \to 1^-$ ($h \to h_\mathrm{m}^-$) we have $C_1 \geq 0$, and we note that the singularities in (13)–(15) as $s \to 1$ are integrable if $C_1 > 0$.

4 Asymptotic Results

In the limit of strong heating $M \to \infty$ equation (14) can be satisfied only if the solution for h_m satisfies $h_\mathrm{m} \to \infty$ and $h_\mathrm{m} = o(M)$. The integrals in (13)–(15) are dominated by global contributions with integrands $s^{1/2}(-3Mh_\mathrm{m} \log s)^{-1/2}$ and $(-3Mh_\mathrm{m} s \log s)^{1/2}$; we thus find that at leading order h_m and a are given by

$$h_\mathrm{m} \sim \left(\frac{9M}{8\pi}\right)^{\frac{1}{3}}, \quad a \sim \left(\frac{\pi}{\sqrt{3}M}\right)^{\frac{1}{3}} \tag{21}$$

as $M \to \infty$, showing that $h_\mathrm{m} \to \infty$ and $a \to 0$ in this limit. Moreover (13) shows that at leading order the free-surface profile is given by

$$h \sim h_\mathrm{m} \exp\left(-2\left[\mathrm{erf}^{-1}\left(\frac{x}{a}\right)\right]^2\right), \tag{22}$$

where erf^{-1} denotes the inverse of the error function.

In the limit of strong cooling $\overline{M} = -M \to \infty$ the maximum height h_m must be finite for the integral in (14) to be real, but equation (14) can be satisfied at leading order only if $C_1 = 0$ in (20). This determines h_m at leading order and a is then determined at leading order from (17); we thus find that

$$h_m \sim \frac{\theta^2}{3\overline{M}}, \qquad a \sim \frac{3\overline{M}}{2\theta^2} \tag{23}$$

as $\overline{M} \to \infty$. Moreover (13) shows that the free-surface profile is flat, with $h \sim h_\mathrm{m}$, except in a boundary layer near $x = a$.

Ehrhard and Davis [2, Eq. 7.8p] solved (10) numerically and found that in the case $G = 0$ and $\theta_\infty \neq 0$, the equilibrium value of a, denoted by a_∞, is given by

$$a_\infty \sim \frac{1.48\overline{M}}{\theta_\infty^2} \tag{24}$$

as $\overline{M} \to \infty$. Our leading order results (23) confirm the form of (24) and show that the numerically determined factor 1.48 should actually be $3/2$. Moreover, our results show that in the limit $\overline{M} \to \infty$ the corrected version of (24) is valid for *all* values of G and, when a_∞ is replaced by a and θ_∞ is replaced by $\theta \neq 0$, is valid for *all* values of t.

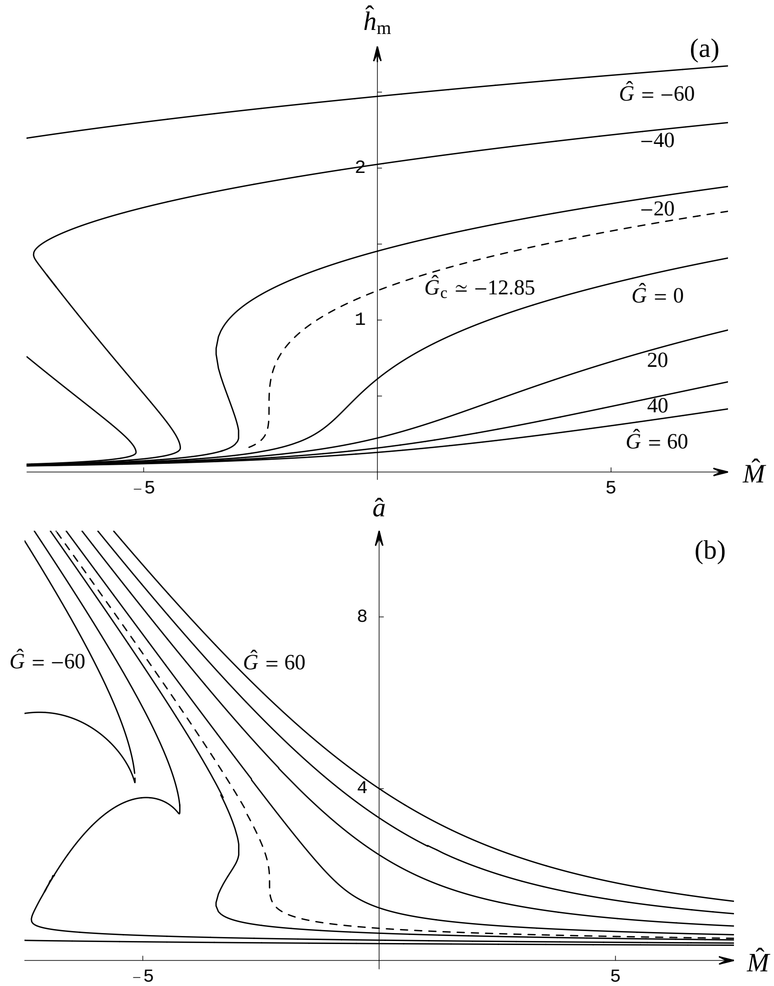

Figure 2: Variation of $\hat{h}_{\rm m} = h_{\rm m}/\sqrt{\theta}$ and $\hat{a} = a\sqrt{\theta}$ plotted as functions of $\hat{M} = M/\theta^{3/2}$ for a range of values of $\hat{G} = G/\theta$.

Similarly, Ehrhard and Davis [2, Eq. 7.5p] showed that for $\theta = G = 0$, a is given by

$$a^3 M = k^3, \tag{25}$$

for all values of M, where k was found numerically to be approximately 1.22. When $\theta = G = 0$, evaluating the integrals in (14) and (15) yields

$$h_{\rm m} = \left(\frac{9M}{8\pi}\right)^{\frac{1}{3}}, \quad a = \left(\frac{\pi}{\sqrt{3}M}\right)^{\frac{1}{3}}, \tag{26}$$

for *all* values of M. This shows that $k = (\pi/\sqrt{3})^{1/3} \simeq 1.2195$ confirming the numerical value given by Ehrhard and Davis [2].

5 Discussion

By re-scaling the variables as follows:

$$\hat{h}_{\rm m} = \frac{h_{\rm m}}{\sqrt{\theta}}, \quad \hat{a} = a\sqrt{\theta}, \quad \hat{M} = \frac{M}{\theta^{\frac{3}{2}}}, \quad \hat{G} = \frac{G}{\theta} \tag{27}$$

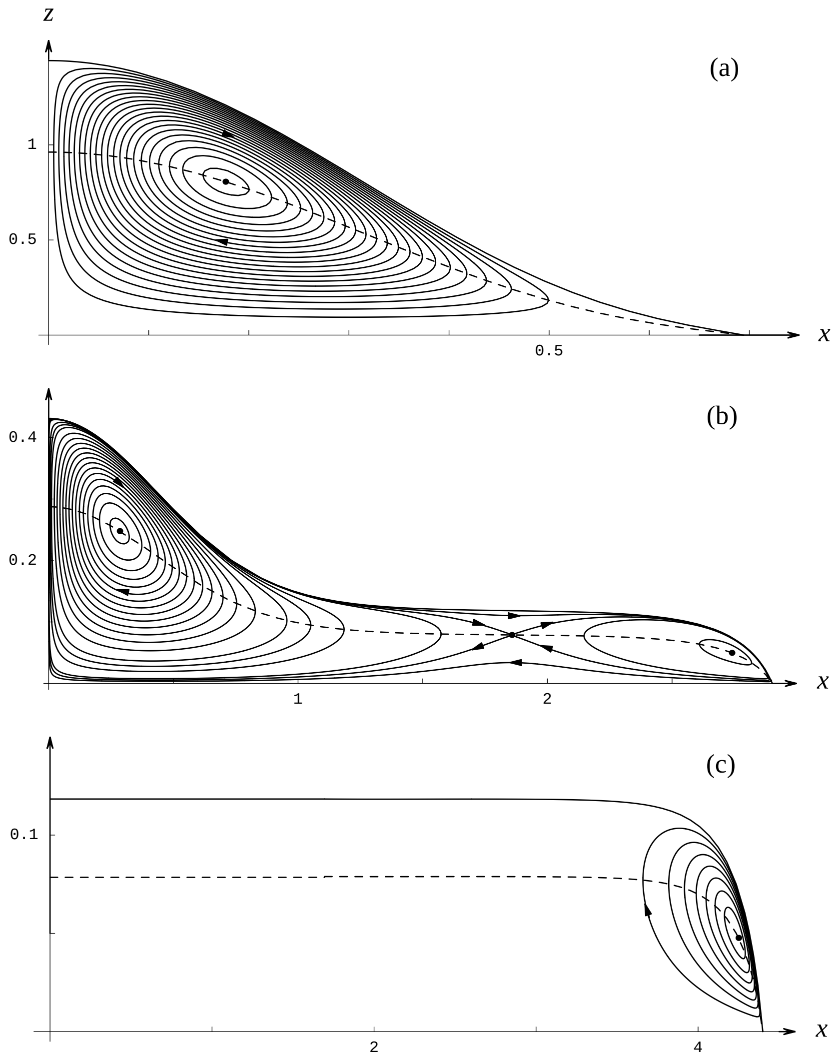

Figure 3: The three possible solutions for the drop profiles and streamlines for the case $\hat{G} = -30$ and $\hat{M} = -4$. The corresponding maximum heights are (a) $\hat{h}_{\mathrm{m}} \simeq 1.45$, (b) $\hat{h}_{\mathrm{m}} \simeq 0.43$, (c) $\hat{h}_{\mathrm{m}} \simeq 0.12$. The dashed curves $z = 2h/3$, which the streamlines cross vertically, and the stagnation points are also shown. Note that in (b) the streamlines are not plotted at equal intervals in ψ.

(for $\theta \neq 0$) we are able to remove explicit reference to θ from the problem, that is, in terms of the hatted variables in (27), the solution is again given by (13)–(16) but with θ set to unity.

Figure 2 shows the variation of the re-scaled maximum height $\hat{h}_{\mathrm{m}}$ and the re-scaled semi-width $\hat{a}$ with the re-scaled Marangoni number $\hat{M}$ for selected values of the re-scaled Bond number $\hat{G}$. Figure 2(b) is in good agreement with Ehrhard and Davis [2, Fig. 8(b)] who show a_∞ as a function of M for the particular case $G = 0$ and $\theta_\infty = 0.5$. Figure 2(b) extends the previous results to all time t and shows that the behaviour given by Ehrhard and Davis [2, Fig. 8(b)] for $G = 0$ is qualitatively correct for all $\theta_\infty \neq 0$. Figure 2 also shows that there exists a critical value $\hat{G}_{\mathrm{c}} \simeq -12.85$ such that for $\hat{G} < \hat{G}_{\mathrm{c}}$, $\hat{h}_{\mathrm{m}}$ and $\hat{a}$ are triple-valued functions of $\hat{M}$ in some interval $\hat{M}_1 \leq \hat{M} \leq \hat{M}_2 < 0$, but are single-valued elsewhere. In other words, when $\hat{G} < \hat{G}_{\mathrm{c}}$ there is a range of values of $\hat{M}$ corresponding to an appropriately cooled substrate in which there are *three* possible drop solutions with the same volume. Examples of these three solutions in the case $\hat{G} = -30$ and $\hat{M} = -4$ are shown in Figure 3.

By evaluating the streamfunction (7) we see that the three possible drop profiles have very different flow patterns (see Figure 3). Figure 3(a) is typical of the case when there is one stagnation point. Here the flow comprises a single closed eddy, with all particles circulating round the stagnation point. On the other hand, the solution in Figure 3(b) has three stagnation points, namely a saddle stagnation point between two "elliptic" stagnation points, all lying on the curve $z = 2h/3$. The streamlines are closed curves, but the flow comprises two internal eddies which in turn are surrounded by circulating fluid. The solution shown in Figure 3(c) also has one stagnation point, with one eddy. However, in this case the flow is essentially confined to a region near the contact line $x = a$. This latter behaviour is typical of solutions in the limit of strong cooling $M \to -\infty$ described earlier.

It is natural to question the stability of such drop profiles. A preliminary linear stability analysis based on (18) suggests that solutions with negative gradient in Figure 2(a) (i.e. with $\mathrm{d}\hat{h}_{\mathrm{m}}/\mathrm{d}\hat{M} < 0$) are always unstable (i.e. the solution shown in Figure 3(b) is unstable).

6 Conclusion

We re-visited the problem of the spreading of a thin two-dimensional drop of incompressible Newtonian fluid on a uniformly heated or cooled horizontal substrate when thermocapillary effects are significant. The work built on that by Ehrhard and Davis [2] who derived the non-linear partial differential equation governing the free-surface profile of the drop. By adapting the method of Holland *et al.* [3] we obtained the (implicit) exact solution of the two-dimensional equation in the limit of quasi-steady motion. We then considered the behaviour of the solution in various asymptotic limits which confirmed and extended some results of Ehrhard and Davis [2]. We showed that multiple solutions are possible for the case of a pendent drop on an appropriately cooled substrate; the three solutions were shown to be qualitatively different in both shape and flow pattern.

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