Paper to be Submitted for Presentation in ECS2005
The European Coating Symposium 2005

Film Formation Model of Shear Thinning Power Law Fluids Using Lubrication Analysis

R W Hewson, N Kapur *and* ***P H Gaskell***

School of Mechanical Engineering, University of Leeds, Leeds. LS2 9JT. UK.

Abstract

Film forming of a shear thinning fluid on a moving substrate is investigated theoretically and comparison made with existing experimental data. The model is based on lubrication theory for a power-law fluid resulting in a set of four ordinary differential equations that are solved as a boundary value problem. Predictions of coating gap to film thickness ratio and residual fluid fraction are found to be in good agreement with experimental data for Capillary numbers less than 1, with agreement for Capillary numbers less than 0.2 being particularly close.

1 Introduction

The process of forming a continuous thin liquid layer on a moving substrate has a range of applications, including coating where deposition of a uniform liquid film is required. Previous analyses for Newtonian fluids includes the work of Landau-Levich [1], Bretherton [2], Ruschaks empirical equation [3] and the Coyne & Elrod model [4]. The coating of a thin liquid film is affected by different parameters as outlined by Quéré [5, 6], such as a range of viscous, surface tension and inertial forces. This paper focuses on flow dominated by surface tension and viscous forces, with inertial forces having negligible effects as is the case of the Newtonian fluid forming investigations listed above.

Coating fluids regularly exhibit non-Newtonian viscous behaviour, leading to a requirement to be able to analyse such flows. Gutfinger & Tallmadge [7] analysed the process of withdrawing a shear thinning fluid fluid, obeying the power law [8], from a large bath. Weinstein & Ruschak [9], obtained a semi-empirical expression, using finite element data, for the withdrawal of a substrate from a fluid filled gap. Their resulting expression:

$$H_\infty = [K(n)R]^{\frac{3}{2n+1}} \left[\frac{\lambda U_{\text{substrate}}}{\sigma}\right]^{\frac{2}{2n+1}}, \quad K(n) = 2.553e^{-0.65n}, \tag{1}$$

has the same form as Gutfinger & Tallmadge's with their gravity dependent term replaced by the empirical variable $K(n)$ (λ, is the power law consistency factor, n, the power index and σ, surface tension).

Kamisli & Ryan [10] examined the film forming process for a power law fluid within a tube when displaced by an air bubble forced along its length at a constant speed. This geometry is similar to that encountered in coating processes such as roll coating and knife coating when a moving surface is withdrawn from a fluid filled gap; the main difference, for low capillary number flows, being the change of reference from that of a stationary interface and moving substrate to a stationary substrate and moving interface, as shown in figure 1, which does not effect the validity of the analysis.

Kamisli & Ryan used a method of matched asymptotic expansions, in much the same way as that of Landau & Levich [2] and Bretherton [2], to develop inner and outer solutions,

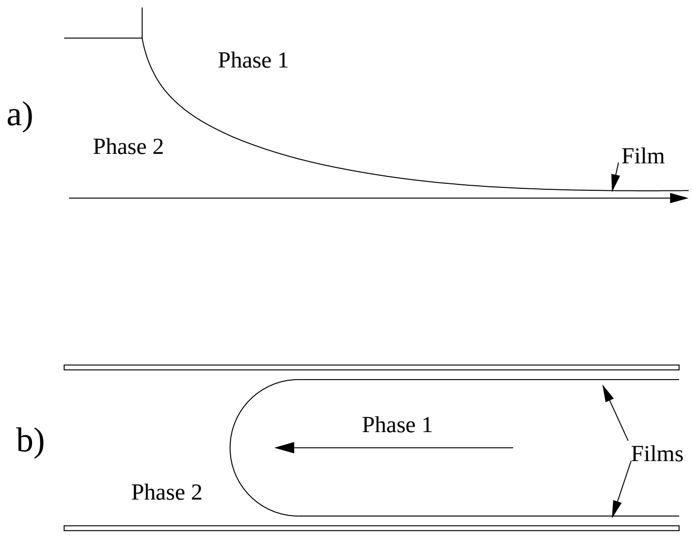

Figure 1: Comparison of coating flows and the gas penetration problem.

incorporating a shear thinning fluid model. They concluded the approach to be inadequate for predicting the film thickness for a power law fluid and attributed this to the lack of accuracy when determining the curvature of the bubble.

Here, the problem is revisited and the film formation of a shear thinning fluid obeying a power law is investigated theoretically and compared with the experimental results of Kamisli & Ryan. The analysis results in a set of four differential equations, which are solved as a boundary value problem (BVP).

2 Lubrication Analysis

Figure 2 defines the coordinate system employed; a local system aligned with the free surface is specified. The flow is assumed to be perpendicular to the free surface rather than the rigid surface sweeping the fluid from the gap. The method used to solve the problem involves a balance of viscous and pressure forces due to the free surface curvature. Making the usual lubrication assumptions we arrive at:

$$\frac{\partial P}{\partial X} = \frac{\partial \tau}{\partial Y}, \tag{2}$$

where τ is the shear stress and P the local pressure. Assuming that the viscosity and/or density of the gas phase is much less than the liquid phase we obtain the following equation describing the shear stress of a generalised Newtonian fluid, based on the boundary condition that $\tau = 0$ at $y = 0$:

$$\tau = \frac{dP}{dX} Y. \tag{3}$$

For a shear thinning fluid which obeys the power law the shear stress is defined as [11],

$$\tau = \lambda \left| \frac{dU}{dY} \right|^{n-1} \frac{dU}{dY}. \tag{4}$$

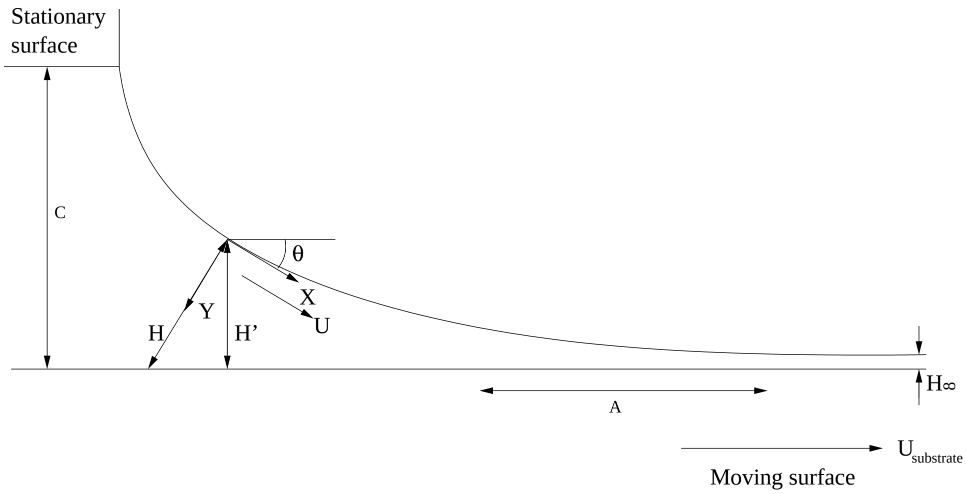

Figure 2: Schematic of the film forming geometry analysed in the BVP for non-Newtonian fluids obeying a power law

Equating the right hand sides of equations (3) and (4) gives, in non-dimensional form:

$$\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy}=\frac{dp}{dx}y, \tag{5}$$

the following non-dimensional scalings having been used throughout the analysis,

$$u=\frac{U}{U_{\text{substrate}}},\quad (x,y,h,h')=\frac{(X,Y,H,H')}{H_\infty}\quad \text{and}\quad p=\frac{PH_\infty^n}{\lambda U_{\text{substrate}}^n}.$$

From continuity of mass considerations the sign of the pressure gradient is positive, i.e. when $h \geq 1$ then $\frac{dp}{dx} \geq 0$. As $y \leq h$ the equation describing the velocity gradient becomes:

$$\frac{du}{dy}=\left(\frac{dp}{dx}y\right)^{\frac{1}{n}}, \tag{6}$$

which can be integrated with respect to y to obtain the velocity profile perpendicular to the free surface. The velocity of the substrate gives the boundary condition, $u_{y=h}=\cos\theta$ which leads to:

$$u=\cos\theta+\frac{\left(\frac{dp}{dx}y\right)^{\frac{1}{n}}ny-\left(\frac{dp}{dx}h\right)^{\frac{1}{n}}nh}{n+1}. \tag{7}$$

Integrating equation (7) from $y=0$ to h and setting the dimensionless flux, q, equal to 1 leads to:

$$\frac{dp}{dx}=\left(\frac{(2n+1)(h\cos\theta-1)}{nh^{\frac{2n+1}{n}}}\right)^n. \tag{8}$$

Balancing the pressure discontinuity across the interface with the surface tension forces gives:

$$p=-\frac{1}{Ca_{ST}}\frac{d\theta}{dx}, \tag{9}$$

where Ca_{ST} is the capillary number defined as $Ca_{ST}=\frac{\lambda U_{\text{substrate}}^n}{\sigma H_\infty^{n-1}}$. The following geometric relations also apply:

$$\frac{dh'}{dx}=\sin\theta, \tag{10}$$

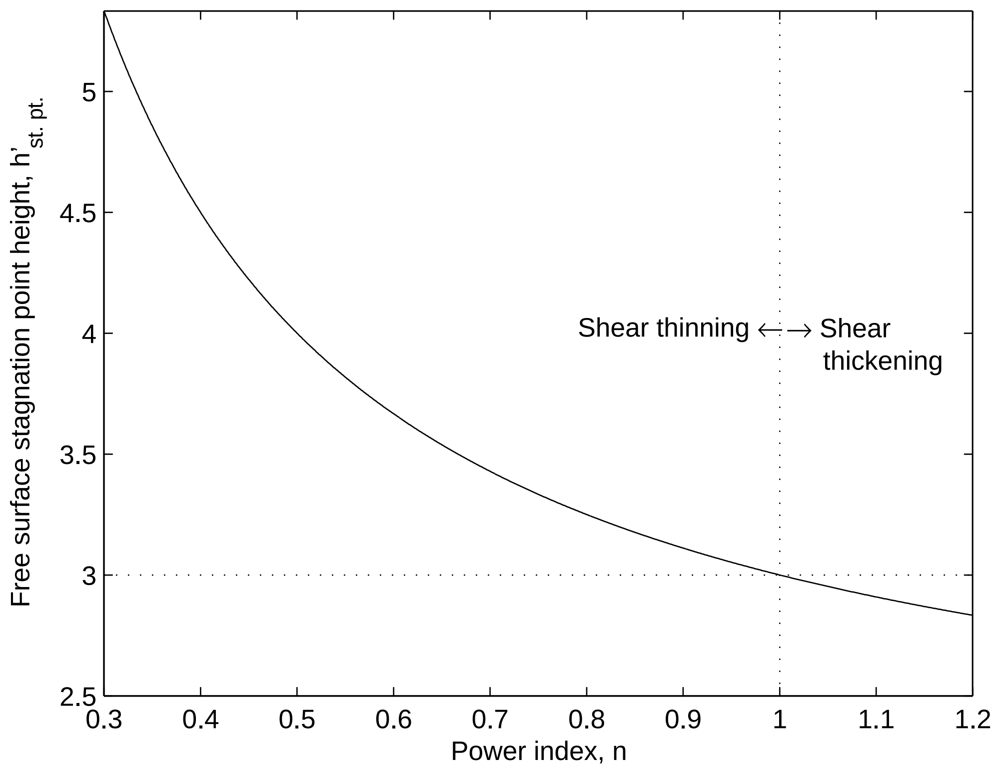

Figure 3: Predicted variation in free surface stagnation point with power law index given by equation (13)

and,

$$\frac{dx'}{dx} = \cos\theta. \tag{11}$$

The four first order ordinary differential equations (8), (9), (10) and (11) describe the film forming process. Equation (11) is not coupled to the other three so can be solved once the solution to equations (8), (9) and (10) has been determined.

Substituting the pressure gradient equation, (8), into the velocity equation, (7) at the free surface, $y = 0$, and after some rearranging, the location of the free surface stagnation point is,

$$h = \frac{2n+1}{n\cos\theta}, \tag{12}$$

which leads to the film height h' at which the stagnation point is located,

$$h' = \frac{2n+1}{n}. \tag{13}$$

This is in clear agreement with the location of the stagnation point predicted by Coyne & Elrod for Newtonian fluids ($n = 1$), where the stagnation point is located at $h' = 3$. The predicted free surface stagnation point is shown in figure 3.

2.1 Boundary Conditions

Four boundary conditions are required to close the problem. These are:

$$p_{x\to\infty} = 0, \tag{14}$$

$$h_{x\to\infty} = 1, \tag{15}$$

$$x'_{x=0} = 0, \tag{16}$$

$$\theta_{x=0} = -\frac{\pi}{2}. \tag{17}$$

Condition (17) can be replaced with $h_{x=0} = c$, where c is gap height, if the contact angle is unspecified but the gap is known. This boundary conditions would be applied in situations where the meniscus is pinned as is often encountered in blade coating.

2.2 Numerical Implementation

Equations (8), (9), (10) and (11) were solved using the BVP solver `BVP4c`, part of the MATLAB package [1]. In addition to the ODEs (8), (9), (10) and (11) the geometric relationship $h = \frac{h'}{\cos\theta}$ was used to relate the solution of equation (10) to equation (8). Using continuation (progressively decreasing the error) a stable calculation method was obtained for power indices greater than 0.5. Each data point could be calculated in less than 3 minutes. Furthermore if an array of points is required the previous solution can be used as an initial guess, greatly reducing computational time.

3 Results

A typical result is shown in figure 4, providing the pressure, free surface velocity and free surface profiles for $n = 0.75$ and $Ca = 0.005$.

The results from the BVP, for the formation of a thin fluid film, are now compared with experimental results obtained for the case of the semi-infinite bubble driven along a tube. For this purpose comparisons are made with the complementary experimental data of Kamisli & Ryan [10]. The capillary number used by Kamisli & Ryan employed the tube radius as the non-dimensionalising length scale as follows:

$$Ca_{K\&R} = \lambda\frac{U^n_{\text{bubble}}}{\sigma R^{n-1}} \equiv Ca_{ST}\frac{H_\infty}{R}^{n-1} \equiv Ca_{ST}h^{n-1}_{x=0} \equiv Ca_{ST}c^{1-n}, \tag{18}$$

where U_{bubble} is the bubble velocity (analogous to the substrate velocity in the BVP analysis) and R is the tube's internal diameter. While the capillary number is based on the final film thickness in the BVP analysis, the capillary number due to Kamisli & Ryan ($Ca_{K\&R}$) is independent of the final film thickness making it easier to compare with experimental data by allowing a comparison of film thickness forming on the inside of the tube rather than a comparison of the tube diameter for a given film thickness.

Agreement with the experimental results of Kamisli & Ryan is very good, see figures 5 and 6, with the effect of decreasing power index leading to a reduction in the final film thickness (or deposited fluid) for capillary numbers less than 0.55. Agreement is best at lower capillary numbers ($Ca_{K\&R} < 0.2$). It should be noted that while the theoretical results of Kamisli & Ryan appear to agree with their experimental data for low capillary numbers they show completely the opposite trend in predicting an increase in residual fluid fraction for increasingly shear thinning fluids (decreasing n) for a given capillary number.

[1] A computational mathematics program from MathWorks.

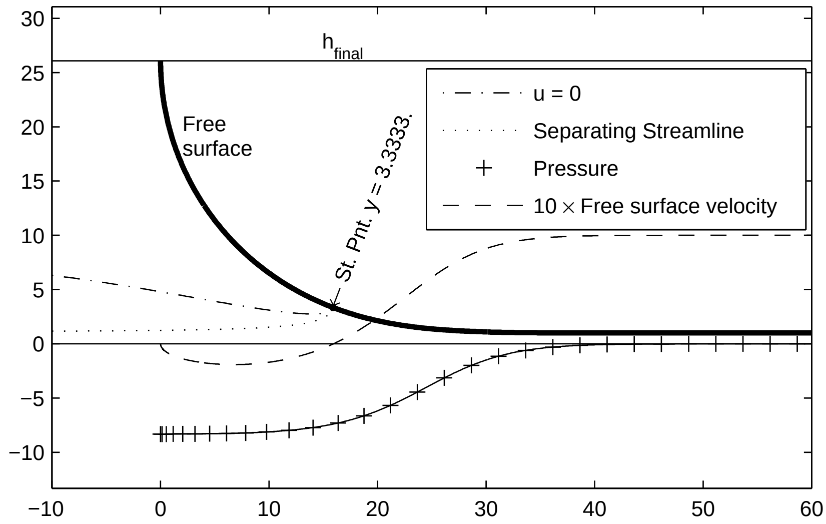

Figure 4: Typical solution for a thin film forming shear thinning fluid, $Ca = 0.005, n = 0.75$.

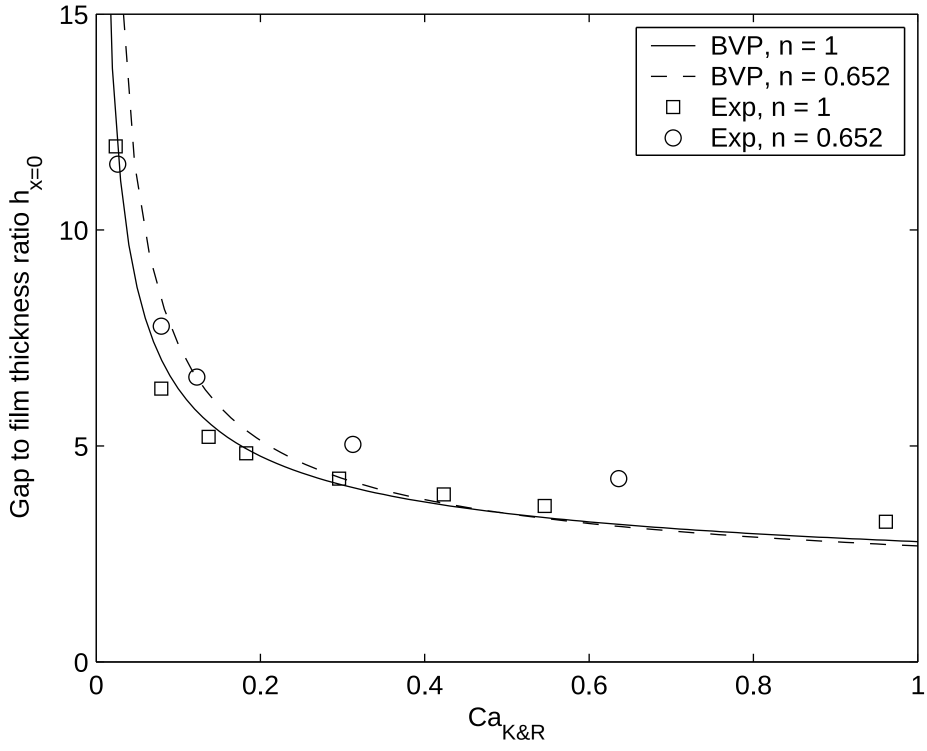

Figure 5: A comparison of prediction with Kamisli & Ryan's [10] tube radius to film thickness ratio for a Newtonian ($n = 1$) and shear thinning ($n = 0.652$) fluids.

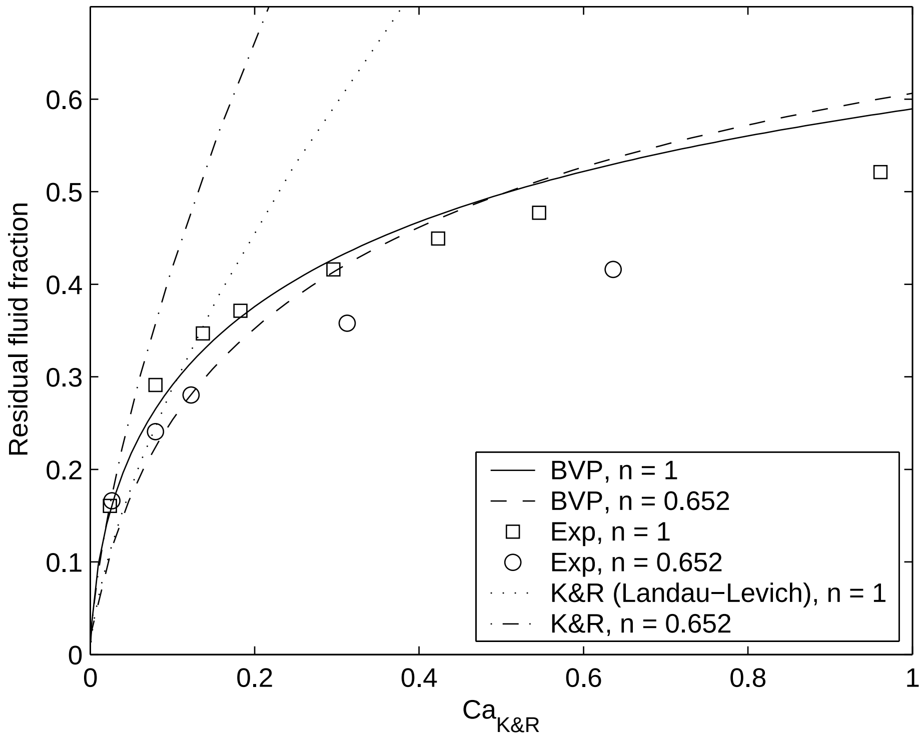

Figure 6: A comparison of prediction with Kamisli & Ryan's [10] residual fluid fraction for a Newtonian ($n = 1$) and shear thinning ($n = 0.652$) fluid.

4 Conclusion

The agreement between the BVP formulated here and the experimental data obtained by Kamisli & Ryan in the course of their investigations is good and provides confidence in relation to applying the model for the solution of a variety of problems involving shear thinning fluids, including many lubrication and coating flows. The conservative limit of $Ca_{K\&R} < 0.2$ to which the BVP solution is applicable has been introduced from a comparison with the experimental data. This work also identifies the need for additional good quality complementary experimental data to be collected for comparison purposes.

References

[1] L. Landau and B. Levich. Dragging of a liquid by a moving plate. *ACTA Physicochimica URSS*, 17:42, 1942.

[2] F.P. Bretherton. The motion of long bubbles in tubes. *Journal of Fluid Mechanics*, 10:166, 1961.

[3] K. J. Ruschak. Boundary-conditions at a liquid air interface in lubrication flows. *Journal of Fluid Mechanics*, 119(JUN):107–120, 1982.

[4] J.C. Coyne and H.G. Elrod. Conditions for the rupture of a lubricating film. part 1: Theoretical model. *Journal of Lubrication Technology*, 92:451, 1970.

[5] D. Quere. Fluid coating on a fiber. *Annual Review of Fluid Mechanics*, 31:347–384, 1999.

[6] D. Quere. Fluid coating. In *European Coating Symposium*, pages 11–32, Strasbourg, 1999.

[7] C. Gutfinger and J. A. Tallmadge. Films of non-Newtonian fluids adhering to flat plates. *Aiche Journal*, 11:403–413, 1965.

[8] J. G. Oldroyd. On the formulation of rheological equations of state. *Proceedings of the Royal Society of London Series a- Mathematical and Physical Sciences*, 200(1063):523–541, 1950.

[9] S. J. Weinstein and K. J. Ruschak. Coating rows. *Annual Review of Fluid Mechanics*, 36:29–53, 2004.

[10] F. Kamisli and M. E. Ryan. Perturbation method in gas-assisted power-law fluid displacement in a circular tube and a rectangular channel. *Chemical Engineering Journal*, 75(3):167–176, 1999.

[11] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids*, volume 1. Wiley, New York, 2 edition, 1987.