Standing waves in viscous film flow down an inclined wavy plane

A. Wierschem *and* **N. Aksel**
Technische Mechanik und Strömungsmechanik, Universität Bayreuth, Germany

We study the flow of a viscous liquid down an inclined channel with a sinusoidal bottom profile of moderate steepness. In experiment, we observed standing waves in resonance with the bottom contour. At the rising edge of the resonance curves, we found humps that are mainly due to second and third harmonics of the bottom undulation. We now propose an analytical approach for these standing waves and humps. Our approach recovers the qualitative features of the experimental observations.

1 Introduction

In many industrial and environmental systems, films of viscous liquid flow along substrates that are usually curved or undulated. Here we focus on gravity-driven films. For monotonously falling bottom contours, the local steady flow of thin gravity-driven films at low Reynolds numbers corresponds essentially to that over a flat incline at the corresponding local inclination angle [1]. At Reynolds numbers of order one, surface waves are generated [1]. But inertia also affects the basic flow: In regimes where inertia becomes significant, hydraulic jumps are generated at the inflow into the flat region of the undulated bottom [2] and for periodic bottom contours even of weak steepness resonant standing waves have been observed [2], [3].

Bontozoglou and co-workers have studied the resonance of viscous film flow down periodic corrugations with capillary-gravity waves. For small amplitudes of the bottom corrugation and rather thick films they found numerically a resonance of the free surface with the bottom contour [4]. At higher steepness, Bontozoglou calculated a skewed, bistable resonance with increasing steepness [5]. While Bontozoglou and co-workers focused on resonance in rather thick films in the capillary and capillary-gravity regime, we found experimentally a resonance in rather thin films in the gravity-wave regime [2]. In this case the film thickness was about the same as the amplitude of the bottom undulation. At the rising edge of the resonance curve we observed humps that form at the flat side of the undulation.

Here, we present an analytical approach to account for the qualitative features of the standing waves as encountered in experiment. For the derivation of a non-linear equation for the film thickness, we employ the Kármán-Polhausen integral boundary-layer method in local coordinates. In this method the velocity profile has to be known *a priori*. Assuming a self-similar parabolic velocity profile, the equation is solved in Cartesian coordinates in a perturbation analysis for the steepness of the bottom contour. In the following section, we derive the equation for the film thickness. It is solved in Section 3 and discussed in Section 4. We finally summarize our conclusions in Section 5.

2 Analytical description of the standing waves

We study the two dimensional film flow of an incompressible Newtonian liquid down a sinusoidal bottom $b(\hat{x}) = a\sin(2\pi\hat{x}/\lambda)$, with wavelength λ, amplitude a, and $\hat{x}$ being the Cartesian coordinate in main flow direction, which is inclined at an angle α with respect to the horizontal, as shown in Figure 1. Considering the wavelength of the bottom variation being much larger than the film thickness, we use an orthogonal curvilinear coordinate system with x being the arc length along the bottom contour and z pointing upwards into the liquid perpendicular to the bottom. This local coordinate system is unambiguous, since the coordinate axes do not cross within the liquid film. The local coordinate system is also shown in Figure 1. The inclination angle due to the bottom profile is $\theta = \arctan b'(\hat{x})$ and the local curvature κ is defined by

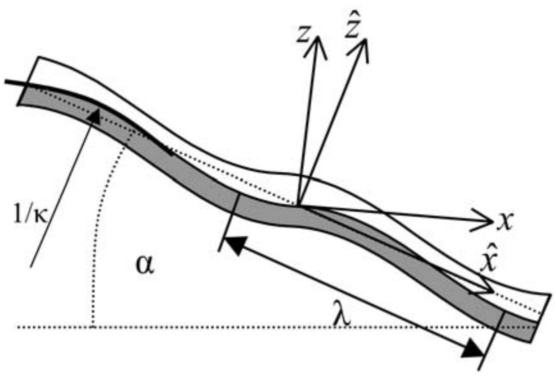

Figure 1: Film flowing down an undulated bottom profile with wavelength λ and mean inclination angle α. The Cartesian coordinates $\hat{x}$ and $\hat{z}$ point in mean flow direction and perpendicular to it. The local coordinates x and z are taken in tangential and perpendicular direction of the profile, respectively.

$$\kappa = -\frac{\partial^2 b}{\partial \hat{x}^2} \bigg/ \left[1+\left(\frac{\partial b}{\partial \hat{x}}\right)^2\right]^{3/2}, \tag{1}$$

where b' denotes the differentiation with respect to the corresponding Cartesian coordinate $\hat{x}$. The elemental lengths in x and z direction are $(1+\kappa z)dx$ and dz, respectively.

A detailed derivation of the set of equations in local coordinates has been given in [1]. Denoting the local position of the free surface, the bottom contour, time, pressure, and the velocity components parallel and perpendicular to the bottom by *f, b, t, p, u,* and *w*, respectively, we apply the following 'natural' scaling

$$\begin{aligned}
&x = \frac{\lambda}{2\pi}X \qquad z = hZ \qquad \kappa = \left(\frac{2\pi}{\lambda}\right)^2 a\mathrm{K} \\
&f = hF \qquad b = aB \qquad \Theta = \arctan\left(\frac{2\pi a}{\lambda}\cos\hat{X}\right), \\
&u = \langle u\rangle U \qquad w = \frac{2\pi h}{\lambda}\langle u\rangle W \qquad t = \frac{1}{\langle u\rangle}\frac{\lambda}{2\pi}T \\
&p = \rho\langle u\rangle^2 P
\end{aligned} \tag{2}$$

where the capital letters refer to the dimensionless quantities and ρ, h, and $\langle u\rangle$ are the density, the film thickness and the mean flow velocity, respectively. The film thickness and the mean flow velocity are taken for the corresponding film flow down a flat plane at inclination angle α at the given flow rate, thus

$$h = \sqrt[3]{\frac{3\nu\dot{q}}{g\sin\alpha}} \qquad \langle u\rangle = \frac{\dot{q}}{h} = \frac{gh^2\sin\alpha}{3\nu}, \tag{3}$$

where $\dot{q}$, g, and ν are the flow rate, acceleration of gravity, and the kinematic viscosity, respectively.

Applying this scaling and introducing the Reynolds number as $Re=\langle u\rangle h/\nu$, the Navier-Stokes equations read

$$
\begin{aligned}
&\delta Re\left(\frac{\partial U}{\partial T}+\frac{1}{1+\delta^2\xi \mathrm{K}Z}U\left(\frac{\partial U}{\partial X}+\delta^2\xi \mathrm{K}W\right)+W\frac{\partial U}{\partial Z}\right)\\
&=-\frac{1}{1+\delta^2\xi \mathrm{K}Z}\delta Re\frac{\partial P}{\partial X}+3\frac{\sin(\alpha-\Theta)}{\sin\alpha}+\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\delta^2\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Z^2}\\
&-\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^3}\delta^4\xi Z\frac{\partial \mathrm{K}}{\partial X}\frac{\partial U}{\partial X}+\frac{1}{1+\delta^2\xi \mathrm{K}Z}\delta^2\xi \mathrm{K}\frac{\partial U}{\partial Z}-\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\delta^4\xi^2\mathrm{K}^2U\\
&+\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^3}\delta^4\xi\frac{\partial \mathrm{K}}{\partial X}W+2\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\delta^4\xi \mathrm{K}\frac{\partial W}{\partial X}
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&\delta^2 Re\left(\frac{\partial W}{\partial T}+\frac{1}{1+\delta^2\xi \mathrm{K}Z}U\left(\frac{\partial W}{\partial X}-\xi \mathrm{K}U\right)+W\frac{\partial W}{\partial Z}\right)\\
&=-Re\frac{\partial P}{\partial Z}-3\frac{\cos(\alpha-\Theta)}{\sin\alpha}+\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\delta^3\frac{\partial^2 W}{\partial X^2}+\delta\frac{\partial^2 W}{\partial Z^2}\\
&-\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^3}\delta^5\xi Z\frac{\partial \mathrm{K}}{\partial X}\frac{\partial W}{\partial X}+\frac{1}{1+\delta^2\xi \mathrm{K}Z}\delta^3\xi \mathrm{K}\frac{\partial W}{\partial Z}-\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\delta^5\xi^2\mathrm{K}^2W\\
&-\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^3}\delta^3\xi\frac{\partial \mathrm{K}}{\partial X}U-2\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\delta^3\xi \mathrm{K}\frac{\partial U}{\partial X}
\end{aligned}
. \tag{5}
$$

where we introduced the dimensionless film thickness $\delta = 2\pi h/\lambda$, and the amplitude to film thickness ratio $\xi = a/h$. The continuity equation for incompressible fluids takes the following form:

$$\frac{\partial U}{\partial X}+\frac{\partial}{\partial Z}\left(1+\delta^2\xi \mathrm{K}Z\right)W=0. \tag{6}$$

In integral form mass conservation yields the dimensionless flow rate

$$\dot{Q}=\int_0^F U dZ. \tag{7}$$

Due to the local coordinate system the boundary conditions at the bottom are:

$$U=W=0. \tag{8}$$

At the free surface the kinematic boundary condition reads

$$\frac{\partial F}{\partial T}+U\frac{1}{1+\delta^2\xi \mathrm{K}F}\frac{\partial F}{\partial X}-W=0. \tag{9}$$

The dynamic boundary condition normal to the surface is now

$$
\begin{aligned}
&Re\left(P-P_{Air}\right)+3\frac{1}{Bo}\frac{\left(1+\delta^2\xi \mathrm{K}Z\right)\frac{\partial^2 F}{\partial X^2}-\left[\left(1+\delta^2\xi \mathrm{K}Z\right)^2+2\delta^2\left(\frac{\partial F}{\partial X}\right)^2\right]\xi \mathrm{K}-\delta^2\xi\frac{\partial \mathrm{K}}{\partial X}Z\frac{\partial F}{\partial X}}{\left[\left(1+\delta^2\xi \mathrm{K}Z\right)^2+\delta^2\left(\frac{\partial F}{\partial X}\right)^2\right]^{3/2}}\\
&=2\delta\frac{\partial W}{\partial Z}-\frac{1}{1+\delta^2\xi \mathrm{K}Z}\delta\frac{\partial F}{\partial X}\left\{\frac{1}{1+\delta^2\xi \mathrm{K}Z}\delta^2\left(\frac{\partial W}{\partial X}-\xi \mathrm{K}U\right)+\frac{\partial U}{\partial Z}\right\}
\end{aligned}
, \tag{10}
$$

where $1/Bo = (2\pi l_{Ca}/\lambda)^2/\sin\alpha$ is an inverse Bond number with $l_{Ca}=\sqrt{\sigma/(\rho g)}$ and σ being the Capillary length and the surface tension, respectively. Finally, the dynamic boundary condition tangential to the free surface reads

$$\left\{\left(1+\delta^2\xi \mathrm{K}F\right)^2-\delta^2\left(\frac{\partial F}{\partial X}\right)^2\right\}\left\{\frac{1}{1+\delta^2\xi \mathrm{K}F}\delta^2\left(\frac{\partial W}{\partial X}-\xi \mathrm{K}U\right)+\frac{\partial U}{\partial Z}\right\}$$
$$+\left(1+\delta^2\xi \mathrm{K}F\right)\delta^2 4\frac{\partial F}{\partial X}\frac{\partial W}{\partial Z}=0 \qquad (11)$$

The Kármán-Polhausen integral boundary-layer method has been applied with success to film flow at high Reynolds numbers [6]. Here we use this method to obtain a single equation for the resonance of capillary-gravity waves with the bottom corrugation. Integrating the continuity equation (6) over the film thickness and taking into account (7), (8), and the kinematic boundary condition (9) yields

$$\frac{\partial F}{\partial T}=-\frac{1}{1+\delta^2\xi \mathrm{K}F}\frac{\partial \dot{Q}}{\partial X}\,. \qquad (12)$$

Integrating the momentum equation in Z direction (5) from a position Z in the liquid up to the free surface and inserting the dynamic boundary condition normal to the free surface (10) yields for the pressure

$$\begin{aligned}
ReP(Z) &= ReP_{Air}+3\frac{\cos(\alpha-\Theta)}{\sin\alpha}(F-Z)\\
&-3\frac{1}{Bo}\frac{\left(1+\delta^2\xi \mathrm{K}Z\right)\frac{\partial^2 F}{\partial X^2}-\left[\left(1+\delta^2\xi \mathrm{K}Z\right)^2+2\delta^2\left(\frac{\partial F}{\partial X}\right)^2\right]\xi \mathrm{K}-\delta^2\xi\frac{\partial \mathrm{K}}{\partial X}Z\frac{\partial F}{\partial X}}{\left[\left(1+\delta^2\xi \mathrm{K}Z\right)^2+\delta^2\left(\frac{\partial F}{\partial X}\right)^2\right]^{3/2}}\\
&+\delta^2 Re\left(\begin{aligned}&\int_Z^F\frac{\partial W}{\partial T}dZ+\int_Z^F\frac{1}{1+\delta^2\xi \mathrm{K}Z}U\frac{\partial W}{\partial X}dZ\\ &-\xi \mathrm{K}\int_Z^F\frac{1}{1+\delta^2\xi \mathrm{K}Z}U^2dZ+\frac{1}{2}W^2(F)-\frac{1}{2}W^2(Z)\end{aligned}\right)\\
&+\delta\frac{\partial W}{\partial Z}(F)+\delta\frac{\partial W}{\partial Z}(Z)-\frac{1}{1+\delta^2\xi \mathrm{K}F}\delta\frac{\partial F}{\partial X}\frac{\partial U}{\partial Z}(F)\\
&-\frac{1}{\left(1+\delta^2\xi \mathrm{K}F\right)^2}\delta^3\frac{\partial F}{\partial X}\left(\frac{\partial W}{\partial X}(F)-\xi \mathrm{K}U(F)\right)+\delta^3\xi\frac{\partial \mathrm{K}}{\partial X}\int_Z^F\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^3}UdZ\\
&-\delta^3\int_Z^F\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\frac{\partial^2 W}{\partial X^2}dZ+3\delta^3\xi \mathrm{K}\int_Z^F\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}\frac{\partial U}{\partial X}dZ\\
&+\delta^5\xi\frac{\partial \mathrm{K}}{\partial X}\int_Z^F\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^3}Z\frac{\partial W}{\partial X}dZ+2\delta^5\xi^2\mathrm{K}^2\int_Z^F\frac{1}{\left(1+\delta^2\xi \mathrm{K}Z\right)^2}WdZ\,.
\end{aligned} \qquad (13)$$

where we made use of the continuity equation (6). Inserting the pressure from (13) into the momentum equation in X direction (4) and integrating it now over the film thickness, using integration by parts in the Reynolds term, the no-slip condition at the bottom (8), and the kinematic boundary condition at the free surface (9) and further eliminating $\partial W/\partial Z$ by applying the continuity equation (6), finally, results in

$$
\begin{aligned}
&\delta Re\left(\begin{aligned}
&\frac{\partial}{\partial T}\int_0^F U dZ+\frac{\partial}{\partial X}\int_0^F \frac{1}{1+\delta^2\xi \mathrm{K} Z}U^2 dZ\\
&+\delta^2\int_0^F \frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left(\begin{aligned}&\frac{\partial}{\partial T}\int_Z^F W dZ-\frac{\partial F}{\partial T}W(F)+\int_Z^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}U\frac{\partial W}{\partial X}dZ\\ &-\xi\mathrm{K}\int_Z^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}U^2 dZ+\frac{1}{2}W^2(F)-\frac{1}{2}W^2(Z)\end{aligned}\right)dZ\\
&+\delta^2\int_0^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^2}\xi\frac{\partial \mathrm{K}}{\partial X}ZU^2 dZ+2\delta^2\xi\mathrm{K}\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}UW dZ
\end{aligned}\right)\\
&=3\frac{\sin(\alpha-\Theta)}{\sin\alpha}F-3\delta\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\frac{\cos(\alpha-\Theta)}{\sin\alpha}(F-Z)\right\}dZ\\
&+3\delta\frac{1}{Bo}\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}dZ\frac{\partial}{\partial X}\frac{\left[\begin{aligned}&\left(1+\delta^2\xi\mathrm{K}F\right)\tfrac{\partial^2 F}{\partial X^2}-\delta^2\xi\tfrac{\partial\mathrm{K}}{\partial X}F\tfrac{\partial F}{\partial X}\\ &-\left[\left(1+\delta^2\xi\mathrm{K}F\right)^2+2\delta^2\left(\tfrac{\partial F}{\partial X}\right)^2\right]\xi\mathrm{K}\end{aligned}\right]}{\left[\left(1+\delta^2\xi\mathrm{K}F\right)^2+\delta^2\left(\tfrac{\partial F}{\partial X}\right)^2\right]^{3/2}}-\frac{\partial U}{\partial Z}(0)\\
&+\delta^2\frac{1}{1+\delta^2\xi\mathrm{K}F}\int_0^F\frac{\partial}{\partial X}\left(\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial U}{\partial X}\right)dZ+\delta^2\frac{1}{1+\delta^2\xi\mathrm{K}F}\delta^2\int_0^F\frac{\partial}{\partial X}\left(\frac{1}{1+\delta^2\xi\mathrm{K}Z}\xi\mathrm{K}W\right)dZ\\
&+\delta^2\frac{1}{1+\delta^2\xi\mathrm{K}F}\xi\mathrm{K}U(F)+\delta^2\frac{1}{\left(1+\delta^2\xi\mathrm{K}F\right)^2-\delta^2\left(\frac{\partial F}{\partial X}\right)^2}4\frac{\partial F}{\partial X}\frac{\partial U}{\partial X}(F)\\
&+\delta^2\xi\mathrm{K}\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial U}{\partial Z}dZ+\delta^2\int_0^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^2}\frac{\partial^2 U}{\partial X^2}dZ\\
&+\delta^2\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\frac{1}{1+\delta^2\xi\mathrm{K}F}\frac{\partial U}{\partial X}(F)+\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial U}{\partial X}\right\}dZ\\
&+\delta^2\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}dZ\frac{\partial}{\partial X}\left\{\frac{\partial F}{\partial X}\frac{\partial}{\partial Z}\left(\frac{1}{1+\delta^2\xi\mathrm{K}Z}U\right)(F)\right\}\\
&+\delta^4\frac{1}{\left(1+\delta^2\xi\mathrm{K}F\right)^2-\delta^2\left(\frac{\partial F}{\partial X}\right)^2}4\xi\mathrm{K}\frac{\partial F}{\partial X}W(F)-3\delta^4\xi\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\mathrm{K}\int_Z^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^2}\frac{\partial U}{\partial X}dZ\right\}dZ\\
&-\delta^4\xi\frac{\partial\mathrm{K}}{\partial X}\int_0^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^3}Z\frac{\partial U}{\partial X}dZ-\delta^4\xi\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\frac{\partial\mathrm{K}}{\partial X}\int_Z^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^3}U dZ\right\}dZ\\
&-\delta^4\xi^2\mathrm{K}^2\int_0^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^2}U dZ+\delta^4\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\int_Z^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^2}\frac{\partial^2 W}{\partial X^2}dZ\right\}dZ\\
&+\delta^4\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}dZ\frac{\partial}{\partial X}\left\{\frac{1}{\left(1+\delta^2\xi\mathrm{K}F\right)^2}\frac{\partial F}{\partial X}\frac{\partial W}{\partial X}(F)\right\}+2\delta^4\xi\mathrm{K}\int_0^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^2}\frac{\partial W}{\partial X}dZ\\
&+\delta^4\xi\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\mathrm{K}\frac{1}{1+\delta^2\xi\mathrm{K}F}W(F)+\mathrm{K}\frac{1}{1+\delta^2\xi\mathrm{K}Z}W\right\}dZ\\
&+\delta^4\xi\frac{\partial\mathrm{K}}{\partial X}\int_0^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^3}W dZ-\delta^6\xi\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\frac{\partial\mathrm{K}}{\partial X}\int_Z^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^3}Z\frac{\partial W}{\partial X}dZ\right\}dZ\\
&-2\delta^6\xi^2\int_0^F\frac{1}{1+\delta^2\xi\mathrm{K}Z}\frac{\partial}{\partial X}\left\{\mathrm{K}^2\int_Z^F\frac{1}{\left(1+\delta^2\xi\mathrm{K}Z\right)^2}W dZ\right\}dZ
\end{aligned}
\qquad . \qquad (14)
$$

This equation is still exact. Together with (12), it forms a set of two coupled equations. In the experiments described in [2], the resonance was observed where the film thickness is of same order as the amplitude of the bottom corrugation, i.e. $\xi \approx 1$. The film-thickness parameter δ was small although not much smaller than unity. In the experiments mentioned above, δ was about 0.3, and thus δ^2 is sufficiently smaller than one to serve as a perturbation parameter. Hence, expanding the terms $1/\left(1+\delta^2\xi \mathrm{K} Z\right)^n$ in (12) and (14) in a Taylor series with $\delta^2 \ll 1$ and using the no-slip condition at the bottom (8) yields

$$
\begin{aligned}
&\delta Re\left(\frac{\partial}{\partial T}\int_0^F U dZ+\frac{\partial}{\partial X}\int_0^F U^2 dZ+O\left(\delta^2\right)\right) \\
&=3\frac{\sin\left(\alpha-\Theta\right)}{\sin\alpha}F-3\delta\int_0^F\frac{\partial}{\partial X}\left\{\frac{\cos\left(\alpha-\Theta\right)}{\sin\alpha}\left(F-Z\right)+O\left(\delta^2\right)\right\}dZ \\
&+3\delta\frac{1}{Bo}\left(F\frac{\partial^3 F}{\partial X^3}-\xi\frac{\partial \mathrm{K}}{\partial X}F+O\left(\delta^2\right)\right) \\
&-\frac{\partial U}{\partial Z}\left(0\right)+\delta^2\left[\begin{array}{l} 3\int_0^F\frac{\partial^2 U}{\partial X^2}dZ+4\frac{\partial F}{\partial X}\frac{\partial U}{\partial X}\left(F\right)+F\frac{\partial}{\partial X}\left\{\frac{\partial U}{\partial X}\left(F\right)\right\} \\ +F\frac{\partial}{\partial X}\left\{\frac{\partial F}{\partial X}\frac{\partial U}{\partial Z}\left(F\right)\right\}+2\xi\mathrm{K}U\left(F\right)+O\left(\delta^2\right) \end{array}\right]
\end{aligned}
\quad . \tag{15}
$$

This equation retains the different physical effects such as gravity, viscous stresses, inertia, hydrostatic and capillary pressure at leading order in the perturbation parameter δ^2. The equation can be recasted into first order by renormalization, i.e. the leading-order terms for each physical effect can all be of comparable order. The term for the gravity forcing $3F\sin(\alpha-\Theta)/\sin\alpha$ and the wall shear stress are of order one. With the Reynolds number being of the order $1/\delta$ or higher, the leading-order inertia terms are of the same order as the gravity forcing. The same holds for the leading-order term of the hydrostatic pressure term with $\cos(\alpha-\Theta)/\sin\alpha$ at small inclination angles and for the capillary-pressure term with large inverse Bond numbers.

Besides the wall shear stress, we also retained for the viscous stresses the viscous forces at the free surface, although they are of order δ^2. Since higher harmonics of the bottom wavelength may occur in resonant interaction, this would diminish their scale in X direction. Retaining higher order derivatives to describe dynamics on a shorter scale than that given by the geometrical quantities of the system is also used in other gravity-driven film flow phenomena [6]. In the following we hence retain the higher order derivatives in the viscous forces while neglecting curvature terms of order δ^2, yet the scale of the bottom curvature is independent from the dynamics. However, as we will see later, it turns out that these terms are not of qualitative relevance for the solution.

A specific profile must now be imposed in the theory. The stationary film flow down a flat plane has a parabolic velocity profile and even in the presence of surface waves the flow is well described by a self-similar parabolic velocity profile [7]. Furthermore, thin films down wavy planes also have a self-similar parabolic velocity profile at leading order when the Reynolds numbers is of order one [1]. Therefore, we may assume here a self-similar parabolic velocity profile, which reads

$$
U=\frac{3}{F}\left[\frac{Z}{F}-\frac{1}{2}\left(\frac{Z}{F}\right)^2\right]. \tag{16}
$$

Since we focus on resonance with standing waves, we assume a stationary flow with the dimensionless flow rate equal to one. By doing so, we finally arrive at an equation for a driven oscillator, however, with nonlinear coefficients

$$\begin{aligned}
&\delta^2 F^2 \frac{\partial^2 F}{\partial X^2} + \left[\frac{1}{2} - \delta^2 \left(\frac{\partial F}{\partial X}\right)^2\right] F \\
&+\frac{1}{2}\delta \frac{\cos(\alpha-\Theta)}{\sin\alpha} F^4 \frac{\partial F}{\partial X} - \frac{1}{2}\delta \frac{1}{Bo}\frac{\partial^3 F}{\partial X^3} F^4 - \frac{1}{5}\delta Re F \frac{\partial F}{\partial X} \\
&= \frac{1}{2}\frac{\sin(\alpha-\Theta)}{\sin\alpha} F^4 - \frac{1}{4}\delta \frac{\partial}{\partial X}\left\{\frac{\cos(\alpha-\Theta)}{\sin\alpha}\right\} F^5 - \frac{1}{2}\delta \frac{1}{Bo}\xi \frac{\partial \mathrm{K}}{\partial X} F^4
\end{aligned} \quad . \tag{17}$$

The oscillation is due to the viscous terms at the free surface and the wall shear stress, i.e. the upper line of (17). Damping is due to the first two terms in the second line, i.e. hydrostatic and capillary pressure. The third derivative in the capillary term leads to strong damping at short wavelengths while the hydrostatic pressure damps at larger wavelengths. These two terms are responsible for the capillary-gravity waves. The damping is counteracted by inertia. Resonance is expected where inertia becomes just equal to the damping. The oscillation is driven by the effects in the third line. They are nonlinear functions of the film thickness itself. The first term in the third line is the gravity-driven forcing by the change in the local inclination angle. The second and third ones are due to the change of the hydrostatic and capillary pressures, respectively, that are caused by the undulated bottom.

3 Analytical solution

To solve (17), we transform the equation into Cartesian coordinates and expanding it into a power series of the bottom steepness $\zeta = 2\pi a/\lambda$. Taking into account that $\Theta = \arctan(\zeta B')$ and that the X-coordinate along the bottom is the arc length

$$dX = \frac{\sqrt{1+\zeta^2 B'^2(\hat{X})}}{1+\delta^2 \xi \mathrm{K} Z} d\hat{X} = \sqrt{1+\zeta^2 B'^2} d\hat{X} + O(\delta^2), \tag{18}$$

we arrive at

$$
\begin{aligned}
0 &= \delta^2 F^2 \frac{\partial^2 F}{\partial \hat{X}^2} + \frac{1}{2} F - \delta^2 \left(\frac{\partial F}{\partial \hat{X}} \right)^2 F - \frac{1}{2} F^4 \\
&+ \frac{1}{2} \delta \cot \alpha F^4 \frac{\partial F}{\partial \hat{X}} - \frac{1}{2} \delta \frac{1}{Bo} F^4 \frac{\partial^3 F}{\partial \hat{X}^3} - \frac{1}{5} \delta Re F \frac{\partial F}{\partial \hat{X}} \\
&+ \zeta \left\{ \frac{1}{2} \left(\cot \alpha + \frac{1}{Bo} + \delta \frac{\partial F}{\partial \hat{X}} \right) F^4 \cos \hat{X} - \frac{1}{4} \delta F^5 \sin \hat{X} \right\} \\
&+ \zeta^2 \left\{ \begin{array}{l}
\frac{1}{2} \left[\begin{array}{l} -\delta^2 F^2 \frac{\partial^2 F}{\partial \hat{X}^2} - \frac{1}{2} \delta \cot \alpha F^4 \frac{\partial F}{\partial \hat{X}} + \frac{3}{4} \delta \frac{1}{Bo} F^4 \frac{\partial^3 F}{\partial \hat{X}^3} \\ + \frac{1}{10} \delta Re F \frac{\partial F}{\partial \hat{X}} + \delta^2 \left(\frac{\partial F}{\partial \hat{X}} \right)^2 F + \frac{1}{4} F^4 \end{array} \right] \\
+ \frac{1}{2} \left[\delta^2 F^2 \frac{\partial F}{\partial \hat{X}} - \frac{3}{2} \delta \frac{1}{Bo} F^4 \frac{\partial^2 F}{\partial \hat{X}^2} + \frac{1}{4} \delta \cot \alpha F^5 \right] \sin 2\hat{X} \\
+ \frac{1}{2} \left[\begin{array}{l} -\delta^2 F^2 \frac{\partial^2 F}{\partial \hat{X}^2} - \frac{1}{2} \delta \cot \alpha F^4 \frac{\partial F}{\partial \hat{X}} + \frac{3}{4} \delta \frac{1}{Bo} F^4 \frac{\partial^3 F}{\partial \hat{X}^3} \\ - \delta \frac{1}{Bo} F^4 \frac{\partial F}{\partial \hat{X}} + \frac{1}{10} \delta Re F \frac{\partial F}{\partial \hat{X}} + \delta^2 \left(\frac{\partial F}{\partial \hat{X}} \right)^2 F + \frac{1}{4} F^4 \end{array} \right] \cos 2\hat{X}
\end{array} \right\} \\
&+ \zeta^3 \left\{ \begin{array}{l}
-\frac{3}{8} \left[\frac{1}{2} \cot \alpha + \frac{1}{Bo} + \delta \frac{\partial F}{\partial \hat{X}} \right] F^4 \cos \hat{X} + \frac{1}{8} \delta F^5 \sin \hat{X} \\
-\frac{1}{8} \left[\frac{1}{2} \cot \alpha + 5 \frac{1}{Bo} + \delta \frac{\partial F}{\partial \hat{X}} \right] F^4 \cos 3\hat{X} + \frac{1}{8} \delta F^5 \sin 3\hat{X}
\end{array} \right\} + O\left(\zeta^4\right) \quad , \qquad (19)
\end{aligned}
$$

where we inserted the sinusoidal bottom profile and rearranged the terms according to the steepness and the modes of the trigonometric functions. In the surface-tension term, we inserted the definition of the curvature (1) and made use of the identity $\xi = \zeta / \delta$.

To solve (19), we expand the surface position F into a power series of the steepness. At leading order there is no steepness. Thus, without undulation there should be no resonance. Therefore, we assume that the leading order of the stationary film thickness is constant:

$$
F = F_0 + \zeta F_1\left(\hat{X}\right) + \zeta^2 F_2\left(\hat{X}\right) + \zeta^3 F_3\left(\hat{X}\right) \dots \qquad (20)
$$

With this expansion, the non-linear coupling is avoided and the non-linear equation (19) degenerates into a hierarchy of linear ordinary differential equations. Inserting of (20) into (19) yields for the leading order term $F_0 = 1$, i.e. the stationary film thickness for a flat incline.

At first order we obtain from (19)

$$
\begin{aligned}
0 &= \delta^2 \frac{\partial^2 F_1}{\partial \hat{X}^2} + \frac{1}{2} \delta \cot \alpha \frac{\partial F_1}{\partial \hat{X}} - \frac{1}{2} \delta \frac{1}{Bo} \frac{\partial^3 F_1}{\partial \hat{X}^3} - \frac{1}{5} \delta Re \frac{\partial F_1}{\partial \hat{X}} - \frac{3}{2} F_1 \\
&+ \frac{1}{2} \left(\cot \alpha + \frac{1}{Bo} \right) \cos \hat{X} - \frac{1}{4} \delta \sin \hat{X} \quad . \qquad (21)
\end{aligned}
$$

The inhomogeneities in (21) have the following origins: the cosine term come from the gravity and capillary driven forcing by the change in the local inclination angle and the sine term is due to the curvature of the bottom, which leads to a change of the hydrostatic pressure. Note that (21) is not an oscillator equation anymore, since the nonlinear forcing term $-F^4/2$ in (19) overcompensates the restoring force $F/2$. Thus, the second order derivative in (19) that had been retained from the viscous forces at the free surface and that yielded the oscillatory part in the non-linear equation (17) looses its qualitative significance and merely results in a small quantitative contribution. Thus, the film flow has no intrinsic oscillation frequency and the homogeneous solution is an exponential function in space. Demanding that it has to remain finite for large distances and periodic for a periodic profile yields a zero homogeneous solution.

According to the inhomogeneities, we solve (21) with the following ansatz:

$$F_1 = C_1 \cos \hat{X} + S_1 \sin \hat{X} , \tag{22}$$

where C_1 and S_1 are constant to be determined. Inserting of (22) into (21) yields

$$\begin{aligned} S_1 &= -\frac{\left\{\delta \cot\alpha + \delta\frac{1}{Bo} - \frac{2}{5}\delta Re\right\}\left(\cot\alpha + \frac{1}{Bo}\right) + \delta\left\{\delta^2 + \frac{3}{2}\right\}}{\left\{\delta \cot\alpha + \delta\frac{1}{Bo} - \frac{2}{5}\delta Re\right\}^2 + 4\left\{\delta^2 + \frac{3}{2}\right\}^2} \\ C_1 &= \frac{-\frac{1}{2}\delta\left\{\delta \cot\alpha + \delta\frac{1}{Bo} - \frac{2}{5}\delta Re\right\} + 2\left\{\delta^2 + \frac{3}{2}\right\}\left(\cot\alpha + \frac{1}{Bo}\right)}{\left\{\delta \cot\alpha + \delta\frac{1}{Bo} - \frac{2}{5}\delta \mathrm{Re}\right\}^2 + 4\left\{\delta^2 + \frac{3}{2}\right\}^2} \end{aligned} . \tag{23}$$

At second order (19) yields together with (20), (22) and (23)

$$\begin{aligned} 0 = {} & \delta^2 \frac{\partial^2 F_2}{\partial \hat{X}^2} + \frac{1}{2}\delta \cot\alpha \frac{\partial F_2}{\partial \hat{X}} - \frac{1}{2}\delta\frac{1}{Bo}\frac{\partial^3 F_2}{\partial \hat{X}^3} - \frac{1}{5}\delta Re \frac{\partial F_2}{\partial \hat{X}} - \frac{3}{2}F_2 \\ & -\frac{3}{2}\left(\delta^2+1\right)\left(C_1^2 + S_1^2\right) + \left(\cot\alpha + \frac{1}{Bo}\right)C_1 - \frac{3}{8}\delta S_1 + \frac{1}{8} \\ & + \begin{bmatrix} -\left(\delta^2+3\right)S_1 C_1 + \frac{1}{2}\left\{2\delta\cot\alpha + 2\delta\frac{1}{Bo} - \frac{1}{5}\delta Re\right\}\left(S_1^2 - C_1^2\right) \\ + \left(\cot\alpha + \frac{1}{Bo}\right)S_1 - \frac{7}{8}\delta C_1 + \frac{1}{8}\delta\cot\alpha \end{bmatrix} \sin 2\hat{X} \\ & + \begin{bmatrix} -\frac{1}{2}\left(\delta^2+3\right)\left(C_1^2 - S_1^2\right) + \left\{2\delta\cot\alpha + 2\delta\frac{1}{Bo} - \frac{1}{5}\delta Re\right\}S_1 C_1 \\ + \left(\cot\alpha + \frac{1}{Bo}\right)C_1 + \frac{7}{8}\delta S_1 + \frac{1}{8} \end{bmatrix} \cos 2\hat{X} \end{aligned} . \tag{24}$$

Thus at second order, the inhomogeneities, which come from the coupling of the free surface with the bottom contour and from nonlinearities of the free surface profile, yield a modification of the mean film thickness and second harmonics of the bottom contour. According to the inhomogeneities in (24) we choose the following ansatz:

$$F_2 = M_2 + C_2 \cos 2\hat{X} + S_2 \sin 2\hat{X} , \tag{25}$$

where M_2, C_2, and S_2 are constants to be determined. Inserting of (25) into (24) yields

$$M_2 = -\left(\delta^2+1\right)\left(S_1^{\,2}+C_1^{\,2}\right)+\frac{2}{3}\left(\cot\alpha+\frac{1}{Bo}\right)C_1-\frac{1}{4}\delta S_1+\frac{1}{12}$$

$$S_2=\frac{KM-LN}{K^2+L^2} \quad , \tag{26}$$

$$C_2=\frac{KN+LM}{K^2+L^2}$$

with

$$K=4\delta^2+\frac{3}{2}$$

$$L=\delta\left\{\cot\alpha+4\frac{1}{Bo}-\frac{2}{5}Re\right\}$$

$$M=-\left(\delta^2+3\right)S_1C_1+\frac{1}{2}\delta\left\{2\cot\alpha+2\frac{1}{Bo}-\frac{1}{5}Re\right\}\left(S_1^{\,2}-C_1^{\,2}\right)$$
$$+\left(\cot\alpha+\frac{1}{Bo}\right)S_1-\frac{7}{8}\delta C_1+\frac{1}{8}\delta\cot\alpha \quad , \tag{27}$$

$$N=\frac{1}{2}\left(\delta^2+3\right)\left(S_1^{\,2}-C_1^{\,2}\right)+\delta\left\{2\cot\alpha+2\frac{1}{Bo}-\frac{1}{5}Re\right\}S_1C_1$$
$$+\left(\cot\alpha+\frac{1}{Bo}\right)C_1+\frac{7}{8}\delta S_1+\frac{1}{8}$$

We finally note that at third order, the inhomogeneities results in a modification of the fundamental mode and in a third harmonic of the bottom contour.

4 Discussion

4.1 First order solution

At first order in ζ, we recover the linear resonance. The first order solution for the film thickness (23) shows that there is a resonance if

$$\cot\alpha+\frac{1}{Bo}-\frac{2}{5}Re=0\,. \tag{28}$$

With the mean flow velocity over the flat incline and with the definitions of Reynolds number and of inverse Bond number, (28) reads

$$\langle u\rangle^2=\frac{5}{6}\left(u_G^{\,2}+u_{Ca}^{\,2}\right)\frac{2\pi h}{\lambda}\,. \tag{29}$$

where $u_G=\sqrt{g\left(\lambda/2\pi\right)\cos\alpha}$ is the phase velocity of gravity waves and $u_{Ca}=\sqrt{\left(2\pi/\lambda\right)\left(\sigma/\rho\right)}$ is the phase velocity of capillary waves with the wavelength of the bottom contour in infinitely thick liquid layers. The factor 5/6 is the inverse shape factor and takes into account the liquid viscosity. We remark that expanding $\tanh(2\pi h/\lambda)$ in a Taylor series we recover the resonance with capillary-gravity waves in liquids of finite thickness.

The changes of the hydrostatic and capillary pressures yield a phase shift of $\pi/2$ with respect to the bottom, while the curvature effect results in a phase shift of π. The amplitude of the gravity-driven term is proportional to cotα, thus being important at low inclination angles. At high inclination angles, cotα becomes small and the inverse Bond number changes only weakly with the inclination angle. With δ smaller but not much smaller one, the curvature term yields an amplitude independent from the inclination angle. For thin films at high inclination angles, the resonance disappears at this order and the free surface follows the bottom contour. Figure 2(a)

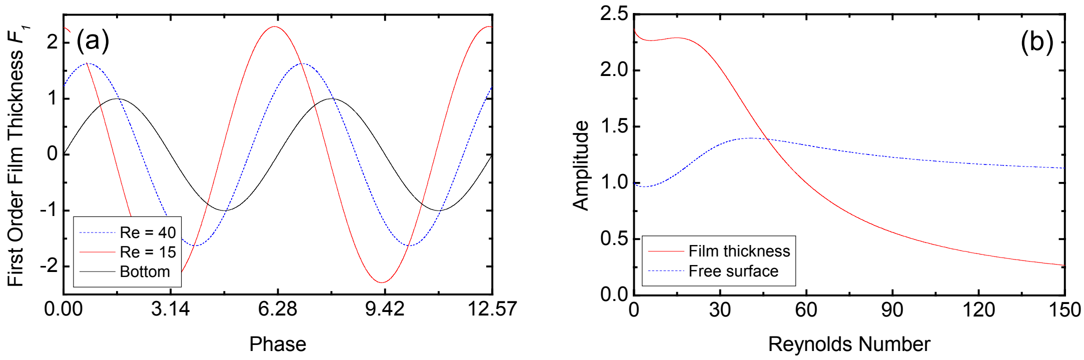

Figure 2: Film thickness at the first order (a) and free surface contour up to the first order (b) along the bottom contour for different Reynolds numbers. *α = 8°, 1/Bo = 0, ν = 200 mm²/s, λ = 300 mm.*

gives an example for the local film thickness variation at first order at different Reynolds numbers. There is a resonance of the film thickness at a Reynolds number of 15, which is $\pi/2$ out of phase with the bottom undulation. With increasing Reynolds number, the phase shift continuously decreases.

For the surface profile one has to take into account the undulation of the bottom itself. In the Cartesian coordinates, the position of the free surface up to first order is at

$$\hat{Z} = \xi B + F_0 + \zeta F_1 = 1 + \xi a_{rel1} \sin\left(\hat{X} + \Delta\varphi_1\right), \tag{30}$$

where we introduced the relative amplitude of the surface contour with respect to the bottom contour

$$a_{rel1} = \sqrt{\left(1 + \delta S_1\right)^2 + \left(\delta C_1\right)^2} \tag{31}$$

and the phase shift between the free surface and the bottom contour

$$\Delta\varphi_1 = \arctan\frac{\delta C_1}{1 + \delta S_1}. \tag{32}$$

It shows that negative S_1 yield a flattening of the free surface while C_1 always produces an increase of the relative amplitude. The corresponding free-surface positions to the film thicknesses of Figure 2(a) are shown in diagram (b) of the same figure.

As appears in Figure 2(a), the phase shift between the film-thickness undulation and the bottom contour decreases with increasing Reynolds number. Beyond the Reynolds number for the maximum amplitude of the film thickness, the diminishing phase shift overcompensates the decline of the film thickness amplitude and yields a maximum undulation of the free surface at higher Reynolds number, as observed in experiments [2].

4.2 Second order solution

The second order solution yields a modification of the mean film thickness and second harmonics of the bottom contour. The mean elevation at second order M_2 reaches its largest values at low Reynolds numbers and tends to zero at high Reynolds numbers. Up to second order, the free surface is at

$$\hat{Z} = \xi B + 1 + \zeta F_1 + \zeta^2 F_2 = \left(1 + \zeta^2 M_2\right) + \xi a_{rel1} \sin\left(\hat{X} + \Delta\varphi_1\right) + \xi a_{rel2} \sin\left(2\hat{X} + \Delta\varphi_2\right) \tag{33}$$

where we introduced the relative amplitude of the second order solution with respect to the bottom contour

$$a_{rel2} = \frac{1}{\xi}\zeta^2 \sqrt{S_2^{\,2} + C_2^{\,2}} \tag{34}$$

and the phase shift

$$\Delta\varphi_2 = \arctan\frac{C_2}{S_2}. \tag{35}$$

Figure 3(a) shows the influence of the second order solution on the surface profiles for a moderately wavy bottom. The higher harmonic of the second order solution produces a large hump at the flat side of the bottom contour, as observed in experiments [2], and another weak one at the steep side. With increasing Reynolds number the hump moves downstream, which is also in agreement with experiments reported in [2]. Figure 3(b) shows the relative amplitude of the surface contour at second order as a function of the Reynolds number together with that of the fundamental from the first order solution. The amplitude of the second harmonic has a maximum at lower Reynolds numbers than that of the fundamental, which is again in agreement with experimental observations [2].

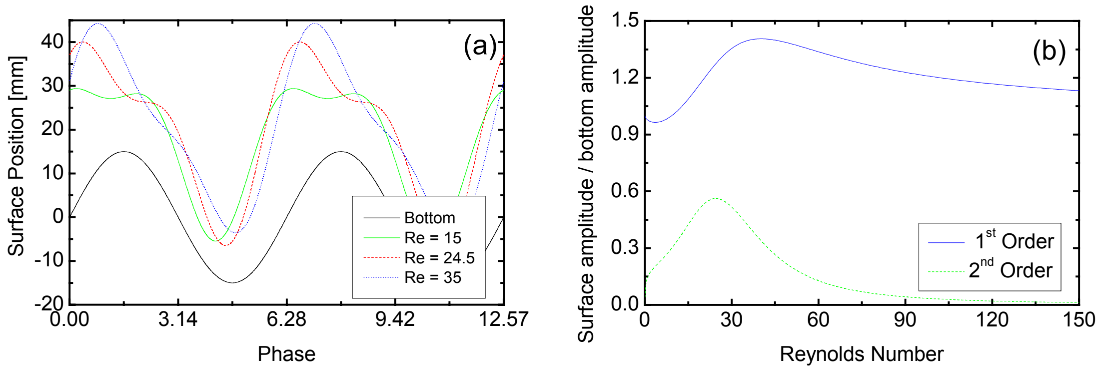

Figure 3: Free surface contour along the bottom contour for different Reynolds numbers (a) and amplitudes of the surface profile (b) including second order terms. $\alpha = 8°$, $1/Bo = 0$, $\nu = 200\ mm^2/s$, $\lambda = 300\ mm$, $a = 15\ mm$.

5 Conclusions

When inertia becomes important in film flow over undulated contours, standing waves with amplitudes larger than the bottom occur. For their qualitative analytical description, we derived a nonlinear equation by applying the integral boundary-layer method in curvilinear coordinates that was solved in Cartesian coordinates with a perturbation approach for the steepness of the bottom. With this approach, we recover the qualitative features of the experimental observations.

For the film thickness, we find a resonance at a mean flow velocity that is about the same as the phase velocity of capillary-gravity waves. Taking into account the displacement due to the bottom undulation, the resonance for the free surface happens at higher mean velocities due to a payoff between decreasing amplitude of the film-thickness undulation and a decreasing phase shift between film-thickness undulation and the bottom contour. Nonlinear resonance yields humps at the flat side of the bottom contour and at lower Reynolds numbers than the resonance of the fundamental in accordance with experiments.

6 References

1. Wierschem, A., Lepski, C. and Aksel N., Effect of long undulated bottoms on thin gravity-driven films, *Acta Mech.* (in press)
2. Wierschem, A. and Aksel, N., Hydraulic jumps and standing waves in gravity-driven flows of viscous liquids in wavy open channels, Phys. Fluids **16** (2004) 3868-3877
3. Vlachogiannis, M. and Bontozoglou, V., Experiments on laminar film flow along a periodic wall, J. Fluid Mech. **457** (2002) 133-156
4. Bontozoglou, V. and Papapolymerou, G., Laminar film flow down a wavy incline, Int. J. Multiphase Flow **23** (1997) 69-79

5. Bontozoglou, V., Laminar film flow along a periodic wall, CMES **1** (2000) 133-142
6. Prokopiou, T., Cheng, M. and Chang, H.-C., Long waves on inclined films at high Reynolds number, J. Fluid Mech. **222** (1991) 665-691
7. Alekseenko, S.V., Nakoryakov, V.Ye. and Pokusaev, B.G., Wave Formation on a Vertical Falling Liquid Film, AIChE **31** (1985) 1446-1461