# Electrical Conductivity Analysis for Networks of Conducting Rods

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## 1. Introduction

In recent year, high aspect-ratio materials, such as metallic nanowires and carbon nanotube, are attractive alternatives for the next generation transparent conductive films. The functionality of the films is represented by opto-electric performance, which mainly affected by the nano- or micro-structures inside the films. In this study, we focus on analyzing the electrical conductivity of two-dimensional networks of conducting rods by treating parts of the networks as the linear circuit system. For the analysis, the multi-nodal representation is used for assigning nodes and edges of the circuit. Based on Kirchhoff's laws, the relation between currents and electrical potentials is formulated using a block matrix equation. After a series of block-matrix-manipulations, the equation can be reduced to yield several simple equations expressed in terms of the incidence matrices and the weighted graph Laplacians. Among them, the equation that represents the Ohm's-law-like relation between the total current and the bias voltage can be used to derive the explicit expression for the normalized conductivity, which can quantify the effect of the network. We performed extensive numerical experiments and statistical tests to confirm the proposed analysis is reasonable for understanding the electrical properties of the networks.

## 2. Incidence matrix for expressing a circuit composed of networks of conducting rods

Figure 1(a) shows an example of the networks of eight conducting rods ($W_1$ to $W_8$). The network is in contact with two external electrodes, which are represented as vertical blue bars

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connected to the external voltage source. These external electrodes and voltage sources form an external circuit. Since the network of conducting rods is biased by a voltage source, we consider the system as an electrical linear circuit. This circuit can be converted to the corresponding incidence matrix by the following procedure.

First, one needs to locate internal and external nodes in the circuit system. Each of the two internal nodes is located on each contact points, which are depicted as green circles, as shown in an inset of the Fig. 1(a). While, external nodes (α, β, and γ) are located on the external electrodes, as shown in purple circles in Fig. 1(a). Second, edges are assigned between nodes. The edges are classified into three categories based on the origin of the resistances: the internal resistance (edges 1 to 10), the junction resistance (edges 11 to 17), and the resistance of the external circuit (edges 18, 19). Based on the nodes and edges indexing, the corresponding incidence matrix is constructed, as shown in Fig. 2. Rows are arranged in order of internal resistance edges, junction resistance edges and external circuit resistance edges; columns are arranged in order of internal nodes and external nodes. The second non-zero component of each row is set to -1 to define the direction of the current flow through a given edge in the circuit system. This incidence matrix is denoted as $\mathbf{G}_{(\varepsilon', \nu')}$.

There are three kinds of networks, as shown in Fig. 1(a): a percolated network (composed of rods $W_1$, $W_2$, and $W_3$), a dangled network (composed of rods $W_4$ and $W_5$) and an isolated network (composed of rods $W_6$, $W_7$, and $W_8$). In this study, the linear circuit is generated from a single cluster of rods that connected through the external electrodes. Therefore, the cluster can consist of percolated networks and dangled networks. Here, the connected component analysis is used for filtering out isolated networks, which do not affect to the conductivity of the circuit. Note that dangled networks do not contribute to the conductivity. However, they are included in the cluster, because it is computationally expensive to select them in the networks of rods. In the language of the graph theory, the overall process can be summarized as extracting subgraph $\mathbf{A'}_{(\varepsilon, \nu+1)}$ from $\mathbf{G}_{(\varepsilon', \nu')}$, where $\mathbf{A'}_{(\varepsilon, \nu+1)}$ is a connected graph possesses external nodes. Figure 1(b) shows the subgraph.

However, the value of rank of a connected graph is always one less than the number of nodes. Therefore, a single column of $\mathbf{A'}_{(\varepsilon, \nu+1)}$ must be removed to guarantee full columns rank of the matrix. In this study, we chose to eliminate the last column, which is a part of the external cirtuit and it is shown in the dotted boxes in Fig. 1(b). The resulting incidence matrix without the dotted boxes is called $\mathbf{A}_{(\varepsilon,\nu)}$. From now on, $\mathbf{A}_{(\varepsilon,\nu)}$ indicates the incidence matrix of the circuit.

Since there are three-types of edges (internal, junction, and external resistances) and two-type of nodes (internal and external nodes), the incidence matrix $\mathbf{A}_{(\varepsilon,\nu)}$ can be partitioned into six block matrices, as shown in Fig. 1(c); $\mathbf{A}_{\boldsymbol{i}(\varepsilon_i,\nu\text{-}2)}$, $\mathbf{A}_{\boldsymbol{j}(\varepsilon_j,\nu\text{-}2)}$, and $\mathbf{A}_{\boldsymbol{e}(\varepsilon_i,2)}$ are incidence matrix related to internal resistance, junction resistance, and edges connected to electrodes, respectively. The total number of edges in the incidence matrix is $\varepsilon = \varepsilon_i + \varepsilon_j + 2$, where $\varepsilon_i$ and $\varepsilon_j$ are the number of internal resistance and junction resistance, respectively.

By using Kirchhoff's law and the partitioned incidence matrix, one can formulate the following matrix equation:

$$\left[\begin{array}{ccc:cc} r_i\mathbf{I}_{(\varepsilon_i,\varepsilon_i)} & \mathbf{0}_{(\varepsilon_i,\varepsilon_j)} & \mathbf{0}_{(\varepsilon_i,2)} & \mathbf{A}_{\boldsymbol{i}(\varepsilon_i,\nu\text{-}2)} & \mathbf{A}_{\boldsymbol{e}(\varepsilon_i,2)} \\ \mathbf{0}_{(\varepsilon_j,\varepsilon_i)} & r_j\mathbf{I}_{(\varepsilon_j,\varepsilon_j)} & \mathbf{0}_{(\varepsilon_j,2)} & \mathbf{A}_{\boldsymbol{j}(\varepsilon_j,\nu\text{-}2)} & \mathbf{0}_{(\varepsilon_j,2)} \\ \mathbf{0}_{(2,\varepsilon_i)} & \mathbf{0}_{(2,\varepsilon_j)} & \mathbf{0}_{(2,2)} & \mathbf{0}_{(2,\nu\text{-}2)} & \mathbf{I}_{(2,2)} \\ \hdashline \mathbf{A}^{\mathbf{T}}_{\boldsymbol{i}(\nu\text{-}2,\varepsilon_i)} & \mathbf{A}^{\mathbf{T}}_{\boldsymbol{j}(\nu\text{-}2,\varepsilon_j)} & \mathbf{0}_{(\nu\text{-}2,2)} & \mathbf{0}_{(\nu\text{-}2,\nu\text{-}2)} & \mathbf{0}_{(\nu\text{-}2,2)} \\ \mathbf{A}^{\mathbf{T}}_{\boldsymbol{e}(2,\varepsilon_i)} & \mathbf{0}_{(2,\varepsilon_j)} & \mathbf{I}_{(2,2)} & \mathbf{0}_{(2,\nu\text{-}2)} & \mathbf{0}_{(2,2)} \end{array}\right] \left[\begin{array}{c} \boldsymbol{i}_{\boldsymbol{i}(\varepsilon_i,1)} \\ \boldsymbol{i}_{\boldsymbol{j}(\varepsilon_j,1)} \\ \boldsymbol{i}_{\boldsymbol{e}(2,1)} \\ \hdashline \boldsymbol{p}_{(\nu\text{-}2,1)} \\ \boldsymbol{p}_{\boldsymbol{e}(2,1)} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{0}_{(\varepsilon_i,1)} \\ \boldsymbol{0}_{(\varepsilon_j,1)} \\ \boldsymbol{b}_{(2,1)} \\ \hdashline \boldsymbol{0}_{(\nu\text{-}2,1)} \\ \boldsymbol{0}_{(2,1)} \end{array}\right],$$

where, $\boldsymbol{i}_{\boldsymbol{i}(\varepsilon_i,1)}$, $\boldsymbol{i}_{\boldsymbol{j}(\varepsilon_j,1)}$, and $\boldsymbol{i}_{\boldsymbol{e}(2,1)}$ are the currents flowing through the edges associated with the internal resistance, the junction resistance, and the external circuit, respectively. $\boldsymbol{p}_{(\nu\text{-}2,1)}$ and $\boldsymbol{p}_{\boldsymbol{e}(2,1)}$ are electric potential of the nodes in the circuit and the external electrodes, respectively. $\boldsymbol{b}_{(2,1)} = \begin{bmatrix} b & 0 \end{bmatrix}^{\mathrm{T}}$, whose first entry represents the bias voltage across the circuit.

Using block-matrix-manipulation, we can obtain the following explicit equations:

$$\boldsymbol{p}_{\boldsymbol{e}(2,1)} = \boldsymbol{b}_{(2,1)},$$

$$\boldsymbol{p}_{(\nu\text{-}2,1)} = -\mathbf{L}^{-1}_{\mathbf{11}(\nu-2,\nu-2)}\mathbf{L}_{\mathbf{12}(\nu-2,2)}\,\boldsymbol{b}_{(2,1)},$$

$$\boldsymbol{i}_{\boldsymbol{e}(2,1)} = \left\{ \mathbf{L}_{\mathbf{22}(2,\,2)} - \mathbf{L}_{\mathbf{21}(2,\,\nu-2)} \mathbf{L}_{\mathbf{11}(\nu-2,\,\nu-2)}^{-1} \mathbf{L}_{\mathbf{12}(\nu-2,\,2)} \right\} \boldsymbol{b}_{(2,1)} , \tag{1}$$

$$\boldsymbol{i}_{\boldsymbol{j}(\varepsilon_j,1)} = \frac{1}{r_j} \mathbf{A}_{\boldsymbol{j}(\varepsilon_j,\,\nu\text{-}2)} \mathbf{L}_{\mathbf{11}(\nu-2,\,\nu-2)}^{-1} \mathbf{L}_{\mathbf{12}(\nu-2,\,2)} \boldsymbol{b}_{(2,1)} ,$$

$$\boldsymbol{i}_{\boldsymbol{i}(\varepsilon_i,1)} = \frac{1}{r_i} \left\{ \mathbf{A}_{\boldsymbol{i}(\varepsilon_i,\,\nu\text{-}2)} \mathbf{L}_{\mathbf{11}(\nu-2,\,\nu-2)}^{-1} \mathbf{L}_{\mathbf{12}(\nu-2,\,2)} - \mathbf{A}_{\boldsymbol{e}(\varepsilon_i,\,2)} \right\} \boldsymbol{b}_{(2,1)} .$$

### 3. Normalized conductivity

From Eq. (1), the total conductivity for the circuit $k_{\mathrm{total}}$ is

$$k_{\mathrm{total}} = \boldsymbol{e}_{(1,2)}^{\mathbf{T}} \left\{ \mathbf{L}_{\mathbf{22}(2,\,2)} - \mathbf{L}_{\mathbf{21}(2,\,\nu-2)} \mathbf{L}_{\mathbf{11}(\nu-2,\,\nu-2)}^{-1} \mathbf{L}_{\mathbf{12}(\nu-2,\,2)} \right\} \boldsymbol{e}_{(2,1)} , \tag{2}$$

where $\boldsymbol{e}_{(1,2)}^{\mathrm{T}} = [1, 0]^{\mathrm{T}}$.

$k_{\mathrm{total}}$ consists only of four components of the weighted graph Laplacian. When the resistances are known *a priori*, the total conductivity of the circuit system can be estimated by Eq. (2). It is interesting to mention that Eq. (2) can be used to determine the total conductivity of metallic nanowire (NW) films even when the microscope images and resistances are available. With a proper image analysis tool, one can construct the incidence matrix from the images. Equation (2) can also be used to estimate the junction resistance when the total conductivity is available, e.g., via a four-point probe method, and the internal resistance of NW is known *a priori*. Note that the junction resistance is hard to measure experimentally.

However, the total conductivity may not be suitable to reflect the network structure of the circuit system. For instance, the total conductivity of a circuit composed of two 0.2 Siemens (S) resistors connected in series is 0.1 S. However, a circuit of ten 0.01 S resistors in parallel yields the same total conductivity. In the former case, the series circuit effectively reduces the conductivity to half of its conductance. In the latter case, the parallel circuit amplifies the conductivity by ten times. From this point of view, the effect of a given network structures may be evaluated by how the networks structure amplifies or reduces the total conductivity of the circuit system from individual conductance (especially minimum one) of the components. The normalized conductivity $k_{\mathrm{N}}$ can represent this effect:

$$k_{\mathrm{N}} = \frac{k_{\mathrm{total}}}{\min(1/r_i\ ,1/r_j)} = k_{\mathrm{total}} \cdot \max(r_i\ , r_j)\ . \tag{3}$$

It reaches to a maximum or minimum value when all the components are connected in parallel or series connection, respectively. Figure 3 shows a demonstration for various circuits and the estimated normalized conductivity.

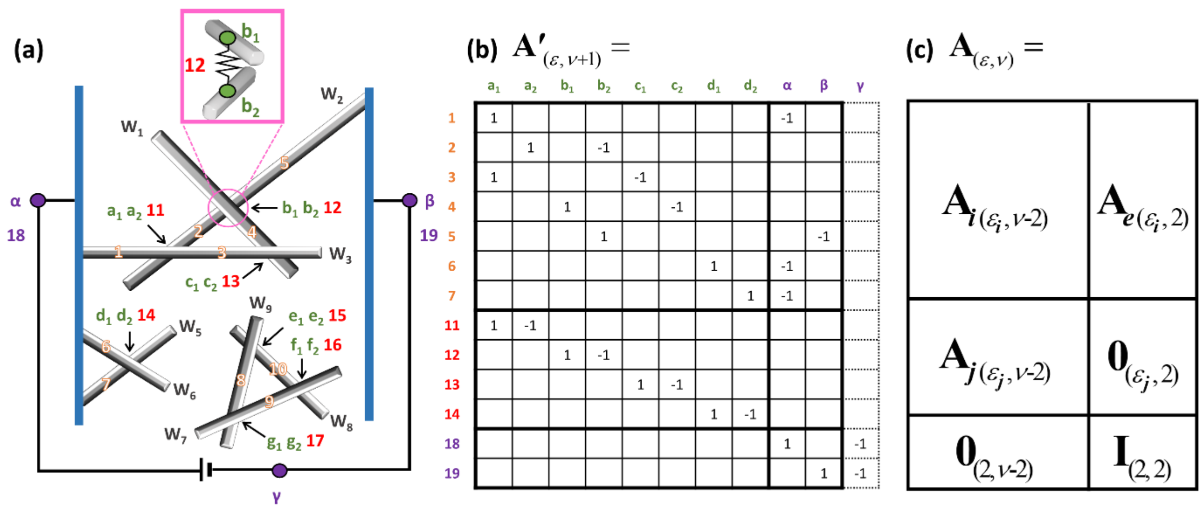
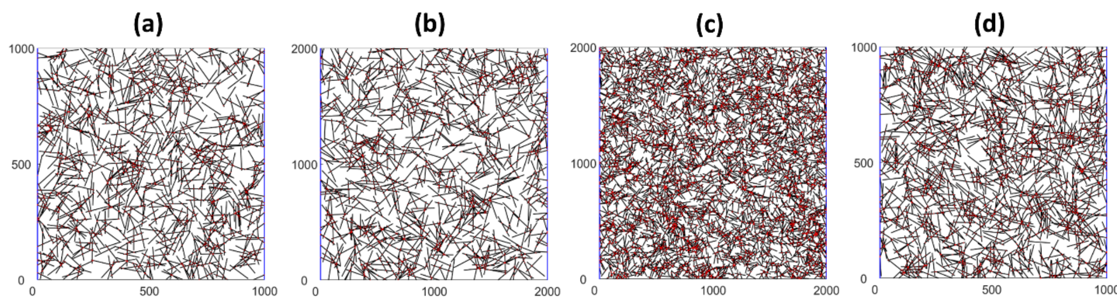



FIG. 1. (a) Example of networks of conducting rods composed of eight rods. Alphabets and numbers denote indices of nodes and edges respectively, where orange, red, and purple are internal, junction, external circuit resistance edges, respectively, while green and purple are internal and external nodes, respectively. (b) the incidence matrix of the circuit in (a), which is the subgraph that connected to the external electrodes. Here, zeros are not denoted for the sake of clarity. (c) The incidence of the circuit omitting the last column of (b), which is the partitioned block matrices.

$$
\mathbf{G}_{(19,17)} =
\begin{array}{c|cccccccccccccc|ccc|}
 & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & d_1 & d_2 & e_1 & e_2 & f_1 & f_2 & g_1 & g_2 & \alpha & \beta & \gamma \\
\hline
1 & 1 & & & & & & & & & & & & & & -1 & & \\
2 & & 1 & & -1 & & & & & & & & & & & & & \\
3 & 1 & & & & -1 & & & & & & & & & & & & \\
4 & & & 1 & & & -1 & & & & & & & & & & & \\
5 & & & & 1 & & & & & & & & & & & & -1 & \\
6 & & & & & & & 1 & & & & & & & & -1 & & \\
7 & & & & & & & & 1 & & & & & & & -1 & & \\
8 & & & & & & & & & 1 & & & & & -1 & & & \\
9 & & & & & & & & & & & 1 & & -1 & & & & \\
10 & & & & & & & & & & 1 & & -1 & & & & & \\
\hline
11 & 1 & -1 & & & & & & & & & & & & & & & \\
12 & & & 1 & -1 & & & & & & & & & & & & & \\
13 & & & & & 1 & -1 & & & & & & & & & & & \\
14 & & & & & & & 1 & -1 & & & & & & & & & \\
15 & & & & & & & & & 1 & -1 & & & & & & & \\
16 & & & & & & & & & & & 1 & -1 & & & & & \\
17 & & & & & & & & & & & & & 1 & -1 & & & \\
\hline
18 & & & & & & & & & & & & & & & 1 & & -1 \\
19 & & & & & & & & & & & & & & & & 1 & -1 \\
\hline
\end{array}
$$

FIG. 2. Constructed incidence matrix $\mathbf{G}_{(\varepsilon',\nu')}$ from Fig. 1(a) containing a percolated network, a dangled network, and an isolated network. Here, zeros are not denoted for the sake of clarity.




**Parameters** (All networks have $r_i/r_j = 1/2000$)

| | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| $l$ | 100 | 200 | 100 | 100 |
| $L/l$ | 10 | 10 | 20 | 10 |
| $n/n_c - 1$ | 0.5 | 0.5 | 0.5 | 1 |

**Results**

| | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| $(\varepsilon, \nu)$ | (5643, 4288) | (5704, 4324) | (22980, 17482) | (10364, 7616) |
| $k_{\mathrm{N}}$ | 0.725 | 0.707 | 0.692 | 1.877 |
| CPU time (sec.) | 2.929 | 3.053 | 155.006 | 14.043 |

FIG. 3. Demonstration of various networks and the conductivity analysis using Eq. (3). For each network, input parameters and the corresponding results are summarized in the top table and bottom table, respectively. Herein, the red dots represent contacts of rods.